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# Bayesian Inference for the Lindley Distribution under Type-II Censoring with Fuzzy Data

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**Abstract:**

This study focuses on estimating the parameters of the Lindley distribution under a Type-II censoring scheme using Bayesian inference. Three approaches—E-Bayesian, hierarchical Bayesian, and Bayesian methods—are employed, with a focus on vague prior data. Accuracy is evaluated using the entropy loss function and squared error loss function (SELF). We assess the efficiency of the proposed methods through Monte Carlo simulations, utilizing the Lindley approximation and Markov Chain Monte Carlo (MCMC). To demonstrate practical applicability, the methodology is applied to a real-world dataset. Comparative results reveal the robustness and accuracy of the approaches. This evaluation underscores the advantages of Bayesian methods in censored parameter estimation, providing insights for reliability analysis and related fields. The study concludes with key findings that support further exploration of Bayesian techniques in censored data analysis. **Keywords:** E-Bayesian Estimation, hierarchical Bayesian estimation, Markov chain Monte Carlo (MCMC), Lindley distribution, Type-II censoring scheme, vague data.

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# 1. Introduction

The Lindley distribution, originally proposed by Lindley (1958), has gained considerable attention due to its ability to model positively skewed lifetime data. Compared to classical models like the exponential distribution, it offers greater flexibility and has been shown to provide a better fit for various real-life reliability datasets (Ghitany et al. (2008); Zakerzadeh and Dolati (2009)). Moreover, its mathematical simplicity and compatibility with Bayesian analysis make it a suitable candidate for modeling under censored and fuzzy data conditions. For more detailed information, Silvey (1967) and Anscombe (1965) can be referred to. Due to its striking similarity to the well-known exponential distribution, it was overlooked for several years. However, Ghitany et al. (2008) extensively explored the statistical properties of this distribution and its application to real datasets. They highlighted its superiority and flexibility in various scenarios compared to the exponential distribution. The estimation of reliability in the Lindley distribution with a progressively Type-II right censored scheme has been discussed by Krishna and Kumar (2011). Gupta and Singh (2013) performed parameter estimation for the Lindley distribution using a hybrid censoring scheme. Singh and Gupta (2012) also studied the load-sharing system model based on the Lindley distribution and applied it to a real dataset. The Lindley distribution, characterized by a parameter $\theta > 0$, has the following probability density function (pdf) and cumulative distribution function (cdf):

$$f(x) = \frac{\theta^2}{\theta + 1}(1 + x)e^{-\theta x}, \tag{1.1}$$

and

$$F(x) = 1 - \frac{1 + \theta(1 + x)}{1 + \theta}e^{-\theta x}. \tag{1.2}$$

Lifetime experiments and reliability studies often encounter a common challenge known as "censoring," where complete information regarding the failure times of all experimental units is not always available to the experimenter. Type-II censoring, which terminates the life testing experiment after the $r^{\text{th}}$ failure (where $r$ is predetermined), is one of the most commonly used censoring schemes. For more comprehensive information, Cohen (1963) and Balakrishnan and Cohen (2014) can be referred to. This censoring scheme has attracted the attention of many scholars. Ng et al. (2006) employed it for parameter estimation of the Birnbaum-Saunders distribution, utilizing both point and distance methods. Singh and Kumar (2007) obtained Bayesian estimation of exponential distribution parameters using the Type-II censoring scheme. Iliopoulos and Balakrishnan (2011) applied it in the context of the Laplace distribution, while Kundu and

Raqab (2012) utilized it for making inferences on the Weibull distribution within a Bayesian framework. Makhdoom et al. (2016) proposed a Bayesian approach to estimate the reliability parameter of the power Lindley distribution, demonstrating its applicability in reliability analysis. Recently, Roodbary and Makhdoom (2024) obtained the parameters of the generalized power Lindley distribution based on the hybrid Type-II censoring scheme.

Statistical modeling plays a crucial role in handling data as it accounts for the inherent randomness present in the data. Its applications span across various scientific disciplines, encompassing numerous continuous variables encountered in our daily lives. When we encounter these variables, they are examined from various perspectives. Although we assume that each measurement of a continuous variable represents an exact value, this assumption is not appropriate due to the inherent nature of continuous phenomena, which cannot be measured with absolute precision. Despite the development of sophisticated tools for precise measurements, the results we obtain are often imprecise and referred to as fuzzy.

Hence, it can be observed that the observed data combines two types of uncertainties: the variation among the observations and the imprecision inherent in each individual observation, known as fuzziness. Some researchers have incorporated fuzzy sets in the estimation theory. Coppi et al. (2006) presented applications of Bayesian methods in a fuzzy framework. Huang et al. (2006) introduced a new method for Bayesian reliability analysis based on fuzzy lifetime data. Akbari and Rezaei (2007) introduced a uniformly minimum variance unbiased point estimator using fuzzy observations. Several studies have been conducted by Pak et al. (2013a), Pak et al. (2014b), Pak (2016), Pak (2017), Pak et al. (2014a), Pak et al. (2013b), Pak and Mahmoudi (2018), and Makhdoom and Pak (2024), focusing on inferential procedures for lifetime distributions based on fuzzy observations.

The hierarchical Bayesian prior distribution was primarily proposed by Lindley and Smith (1972). Subsequently, Han (1997) examined the structure of the hierarchical prior distribution, along with the E-Bayesian method and its applications. In recent years, Han (2009) further investigated the E-Bayesian and hierarchical Bayesian methods for estimating the exponential parameter and the ratio in the binomial distribution. Jaheen and Okasha (2011) derived the E-Bayesian estimation for the Burr type-XII model using a Type-II censoring scheme. Furthermore, Wang et al. (2012) conducted the E-Bayesian estimation and hierarchical Bayesian estimation of system reliability to estimate parameters in the Pascal distribution. More recently, Yaghoobzade Shahrestani and Makhdoom (2021) computed E-Bayesian and hierarchical Bayesian estimations for $R = P(X > Y)$ in the Weibull distribution. Makhdoom et al. (2023) obtained the E-Bayesian and hierarchical Bayesian estimation of reliability in a multi-component stress-strength model based on the inverse Rayleigh distribution. Very recently, Alotaibi et al. (2023) obtained the estimation of the modified Lindley distribution applying a progressive Type-II censoring scheme. Yaghoobzadeh Shahrastani (2019) performed E-Bayesian and hierarchical Bayesian analyses for the scalar parameter of the Gompertz distribution based on fuzzy data under a Type-II censoring scheme. Additionally, Heidari et al. (2022) made inferences on the E-Bayesian and hierarchical Bayesian estimation of the Rayleigh distribution parameter using a Type-II censoring scheme and imprecise data.

In many real-world applications, data collected from observations, surveys, or expert opinions is inherently imprecise or vague. Traditional crisp data models may not adequately capture this uncertainty, leading to reduced accuracy or oversimplified representations of complex situations.

Fuzzy data provides a flexible framework to handle such ambiguity by allowing values to be expressed in degrees of membership rather than as fixed points. This is particularly useful in domains such as decision-making, risk assessment, and human behavior modeling, where subjective judgments and uncertain inputs are common.

Given that no previous attempts have been made to apply a Bayesian approach for estimating parameters in the Lindley model using a Type-II censoring scheme in the presence of fuzzy data, we were inspired to undertake this study. We employ the E-Bayesian and hierarchical Bayesian methods with a Type-II censoring scheme to estimate the parameter $\theta$ using fuzzy data. The estimation process is guided by the entropy loss function, which is defined as follows:

$$L(\hat{\theta}, \theta) \propto \left(\frac{\hat{\theta}}{\theta}\right)^k - k \ln\left(\frac{\hat{\theta}}{\theta}\right) - 1, \quad k \neq 0.$$

In many real-world scenarios, information is not always precise or clearly defined. Traditional data models often rely on crisp values—such as binary classifications or exact numerical measurements—which may not adequately capture the vagueness or uncertainty present in human reasoning, perception, or naturally imprecise environments. To address this, **fuzzy data** provides a mathematical framework for handling imprecision and partial truth.

Fuzzy data is rooted in *fuzzy set theory*, introduced by Zadeh (1965), where an element's membership in a set is expressed by a value between 0 and 1, rather than as a binary (0 or 1) decision. A *fuzzy set* $A$ in a universe of discourse $X$ is defined by a *membership function* $\mu_A : X \to [0, 1]$, where $\mu_A(x)$ denotes the degree to which element $x$ belongs to set $A$. For instance, in describing temperature, the statement "It is hot" cannot be accurately represented by a single threshold; rather, fuzzy data allows “hot” to be a gradual concept, with values like 0.2 (slightly hot), 0.7 (hot), or 1.0 (very hot).

This approach is particularly useful in systems where data is derived from *expert opinions*, *linguistic assessments*, or *imprecise measurements*, such as in medical diagnosis, decision-making systems, and social sciences. In our study, fuzzy data is used to obtain estimates where crisp values would not reflect the underlying uncertainty accurately.

By incorporating fuzzy data, we aim to capture the inherent vagueness in the input information and produce results that are more aligned with real-world behavior and human reasoning.

In the upcoming sections, we will provide a concise overview of some fundamental definitions that are necessary for understanding the subsequent discussions. These definitions were initially introduced by Zadeh (1968) and are presented below.

**Definition 1.1.** *Consider a universal set $X$. We define a fuzzy set denoted as $\tilde{A}$ by utilizing the membership function $\mu_{\tilde{A}}(x) : \mathbb{R} \to [0,1]$. Here, $x \in X$ represents the degree to which $x$ belongs to the fuzzy set $\tilde{A}$. A fuzzy set can be represented as a pair consisting of a set $X$ (which is typically required to be non-empty) and the membership function $\mu_{\tilde{A}}$. The universe of discourse, often denoted by $X$ or $U$, is the reference set. For each $x \in X$, the value $\mu_{\tilde{A}}(x)$ is referred to as the grade of membership of $x$ in the fuzzy set $A$.*

**Definition 1.2.** *A fuzzy set $\tilde{A}$ in a universal set $X$ is considered normal if and only if the supremum of the membership function over $X$, $\sup_{x\in X} \mu_{\tilde{A}}(x)$, equals 1.*

**Definition 1.3.** *A fuzzy set $\tilde{A}$ in a universal set $X$ is defined as convex if the following equation holds for all $x, y \in X$ and $\lambda \in [0,1]$:*

$$\mu_{\tilde{A}}\left(\lambda x + (1-\lambda)y\right) \geq \min\left\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\right\}.$$

**Definition 1.4.** *If $X$ is a universal set and the fuzzy set within $X$ is both normal and convex, then the fuzzy set of $X$ is referred to as a fuzzy number.*

**Definition 1.5.** *Suppose we have two continuous functions $L$ and $R$ defined as $L : \mathbb{R}^+ \to [0,1]$ and $R : \mathbb{R}^+ \to [0,1]$, which possess the following characteristics:*

$$\begin{aligned} &L(-x) = L(x), \quad R(-x) = R(x), \\ &L(0) = R(0) = 1, \end{aligned}$$

*and*

$$\lim_{x\to\infty} L(x) = \lim_{x\to\infty} R(x) = 0.$$

If the functions $L$ and $R$, defined on the interval $[0,\infty)$, are both decreasing, then the fuzzy number $\tilde{M}$ can be classified as a type of $L$-$R$ if the following equation holds:

$$\mu_{\tilde{M}}(x) = \begin{cases} L\left(\frac{m-x}{\alpha}\right) & x \leq m, \quad \alpha > 0, \\ R\left(\frac{m-x}{\beta}\right) & x \geq m, \quad \beta > 0. \end{cases}$$

The average of the fuzzy number $\tilde{M}$, denoted by $m$, is used to determine its left and right bounds, $\alpha$ and $\beta$, respectively. The fuzzy number of type $L$-$R$ is represented as $\tilde{M} = (m, \alpha, \beta)$. The key innovation lies in incorporating fuzzy data into a Type-II censored Bayesian framework, which to the best of our knowledge has not been previously addressed in the context of the Lindley distribution.

The structure of the paper is as follows: Section 2 explains the model in detail. Section 3 of the article is dedicated to E-Bayesian estimation. Hierarchical Bayesian estimations are mentioned in Section 4. Section 5 focuses on simulation studies. We analyze a real dataset in Section 6, and finally, we present the findings and conclusions in Section 7.

## 2. Model definition: problem formulation and assumptions

Bayesian methods are particularly well-suited for analyzing lifetime data under censoring and fuzzy uncertainty. Unlike frequentist techniques, which rely heavily on precise observations and large-sample properties, the Bayesian framework naturally accommodates imprecision by modeling uncertainty through prior distributions. Additionally, Bayesian inference yields full posterior distributions for the parameters, allowing credible intervals and loss-based estimates to be easily derived. These advantages, along with computational feasibility via MCMC techniques, make Bayesian methods the preferred approach in this study.

Upon conducting an experiment to measure the lifetime of $n$ separate units, symbolized as $X_1, X_2, \cdots, X_n$, which correspond to their individual durations of functionality, it is presumed that these units are independent and identically distributed (iid) with a probability density function (pdf), which is elaborated in Eq. (1.1). Before initiating the test, we choose a number $r$, which is less than $n$, with the intention to conclude the test upon the occurrence of the $r^{th}$ failure. We are now faced with a situation where exact failure timings are not recorded, a result of adhering to a Type-II censoring scheme. We are left with incomplete data represented by fuzzy numbers $\tilde{x}_i = (m, \alpha_i, \beta_i)$ for the indices $i$ ranging from

1 to $r$. These fuzzy numbers are each paired with their corresponding membership function $\mu_{\tilde{x_i}}(x_i)$. The largest average of these fuzzy values is represented as $m_{(r)}$. Moreover, we express the lifetime of the remaining $n-r$ units, which are withdrawn post the $r^{th}$ failure, as fuzzy numbers $\tilde{x}_i$ for the indices $i = r+1$ to $n$, each with an associated degree of membership function

$$\mu_{\tilde{x_j}}(x) = \begin{cases} 0 & x \leq m_{(r)}, \\ \\ 1 & x > m_{(r)}, \end{cases} \quad , \quad j = r+1, \cdots n.$$

Suppose we are given a random sample of size $n$ from the Lindley distribution, denoted as $X_1, X_2 \cdots, X_n$, with a probability density function (pdf) as specified in Eq. (1.1). The random vector formed by these variables is denoted as $\mathbf{X} = (X_1, \cdots, X_n)$. To calculate the likelihood function for the complete data, we can express it as follows:

$$L(\theta; \mathbf{x}) = \frac{\theta^{2n}}{(\theta+1)^n} \left( \prod_{i=1}^{n} (1+x_i) \right) \exp(-\theta \sum_{i=1}^{n} x_i).$$

We have a concrete set of outcomes for $\mathbf{X}$ as $\mathbf{x} = (x_1, \cdots, x_n)$ with precisely known observed values. However, the definitive values of $\mathbf{X}$ are not accessible; instead, we have incomplete data represented by a fuzzy set $\tilde{\mathbf{x}} = (\tilde{x}_1, \cdots, \tilde{x}_n)$, accompanied by the membership function $\mu_{\tilde{\mathbf{x}}}(\mathbf{x}) = \mu_{\tilde{x_1}}(x_1) \times \cdots \times \mu_{\tilde{x_n}}(x_n)$. This vector that captures the observed lifetimes is considered as fuzzy data. For calculating the likelihood function of the observed fuzzy data, Zadeh (1968)'s concept of the probability associated with a fuzzy event is applied, leading us to the subsequent theorem:

**Theorem 2.1.** *Let's assume that $X_1, \cdots, X_n$ is a sample set derived from the Lindley distribution. For such a dataset, the likelihood function applicable to the fuzzy data is based on the Type-II censoring scheme and is articulated as follows:*

$$\begin{aligned} \ell(\theta; \tilde{\mathbf{x}}) = & \frac{\theta^{2r}}{(1+\theta)^n} \left(1 + \theta(1 + m_{(r)})\right)^{n-r} \exp(-(n-r)m_{(r)}\theta) \\ & \times \left( \prod_{j=1}^{r} \int_0^{\infty} (1+x) e^{-\theta x} \mu_{\tilde{x_j}}(x) dx \right). \end{aligned} \tag{2.3}$$

*Proof.* Referring to the work of Zadeh (1968), we can derive the likelihood function for fuzzy data in the following manner:

$$
\begin{aligned}
\ell(\theta;\tilde{\mathbf{x}}) &= \int_0^\infty f(x;\theta)\mu_{\tilde{\mathbf{x}}}(x)\,dx \\
&= \prod_{j=1}^{n}\int_0^\infty \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\mu_{\tilde{x}_j}(x)\,dx \\
&= \left(\prod_{j=1}^{r}\int_0^\infty \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\mu_{\tilde{x}_j}(x)\,dx\right) \\
&\quad\times\left(\prod_{j=r+1}^{n}\int_0^\infty \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\mu_{\tilde{x}_j}(x)\,dx\right) \\
&= \left(\prod_{j=1}^{r}\int_0^\infty \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\mu_{\tilde{x}_j}(x)\,dx\right) \\
&\quad\times\left(\prod_{j=r+1}^{n}\left[\int_0^{m_{(r)}} \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\mu_{\tilde{x}_j}(x)\,dx + \int_{m_{(r)}}^\infty \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\mu_{\tilde{x}_j}(x)\,dx\right]\right) \\
&= \left(\prod_{j=1}^{r}\int_0^\infty \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\mu_{\tilde{x}_j}(x)\,dx\right) \\
&\quad\times\left(\prod_{j=r+1}^{n}\int_{m_{(r)}}^\infty \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\mu_{\tilde{x}_j}(x)\,dx\right). \qquad (2.4)
\end{aligned}
$$

By performing straightforward calculations and solving two integrals, we arrive at Eq. (2.3) and successfully complete the proof. □

## 3. Estimating the E-Bayesian

Since the Gamma distribution offers considerable flexibility due to its two-parameter structure, it can model a wide range of prior beliefs depending on the values of the shape ($a$) and scale ($b$) parameters. Moreover, it serves as a conjugate prior for several likelihood functions, which simplifies analytical derivations in Bayesian inference. Therefore, we assume that the prior distribution of $\theta$ follows a Gamma distribution with the following probability density function (pdf):

$$
\pi(\theta|a,b) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}, \quad \theta>0,\ a,b>0. \qquad (3.5)
$$

According to Han (1997), if the prior distribution of $\theta$ follows a Gamma distribution with shape parameter $a$ and scale parameter $b$, the derivative of the probability density function $\pi(\theta|a,b)$ with respect to $\theta$ can be expressed as:

$$\frac{d\pi(\theta|a,b)}{d\theta} = \frac{b^a\theta^{a-2}e^{-b\theta}}{\Gamma(a)}\left((a-1)-b\theta\right),$$

where $b>0$ and $0<a\leq 1$. This equation reveals that the rate of change of the pdf with respect to $\theta$ is negative, indicating a decrease in the pdf as $\theta$ increases. According to Berger (2013), increasing the scale parameter $b$ in the Gamma prior distribution for $\theta$ leads to a decrease in the efficiency of the Bayesian estimator for $\theta$. Therefore, it is advisable to impose an upper bound on the value of $b$, such that $0<b<c$. Additionally, Han (2011) suggested that a suitable choice for modeling the distribution of $b$ is a uniform distribution. In this study, we assume that the distribution of the rate parameter $b$ follows a continuous uniform distribution on the interval $(0,c)$, denoted as $\pi_1(b)$. With this assumption, the probability density function for $\theta$ given $b$ can be expressed as follows:

$$\pi(\theta|b) = be^{-b\theta}. \tag{3.6}$$

Equations (2.3) and (3.6) present the prior distribution for handling fuzzy data as follows:

$$\pi(\theta|\tilde{\mathbf{x}}) = \frac{\frac{\theta^{2n}}{(1+\theta)^n}e^{-\theta\left[b+(n-r)m_{(r)}\right]}\left[\prod_{j=1}^{r}\left(\int_0^\infty(1+x)e^{-\theta x}\mu_{\tilde{x_j}}(x)dx\right)\right]}{\int_0^\infty\frac{\theta^{2n}}{(1+\theta)^n}e^{-\theta\left[b+(n-r)m_{(r)}\right]}\left[\prod_{j=1}^{r}\left(\int_0^\infty(1+x)e^{-\theta x}\mu_{\tilde{x_j}}(x)dx\right)\right]d\theta}. \tag{3.7}$$

The Bayesian estimation of a function of $\theta$, denoted as $h(\theta)$, using the entropy loss function can be expressed as follows:

$$\begin{aligned}\hat{U}^h_{Bay}(b) &= \left\{E_{\theta|\tilde{\mathbf{x}}}(h(\theta))^{-k}\right\}^{-\frac{1}{k}}\\ &= \left\{\frac{\int_0^\infty(h(\theta))^{-k}\frac{\theta^{2n}}{(1+\theta)^n}e^{-\theta\left[b+(n-r)m_{(r)}\right]}u(\theta)d\theta}{\int_0^\infty\frac{\theta^{2n}}{(1+\theta)^n}e^{-\theta\left[b+(n-r)m_{(r)}\right]}u(\theta)d\theta}\right\}^{-\frac{1}{k}},\end{aligned} \tag{3.8}$$

where

$$u(\theta) = \prod_{j=1}^{r}\left(\int_0^\infty(1+x)e^{-\theta x}\mu_{\tilde{x_j}}(x)dx\right).$$

In most cases, it is not feasible to calculate it directly. Therefore, the Lindley approximation, introduced by Lindley in 1980, is often employed to approximate its value. The Lindley approximation assumes the following:

$$w(\theta) = \ln l(\tilde{\mathbf{x}},\theta) + \ln\pi(b) = L(\theta)+v(\theta), \tag{3.9}$$

where

$$L(\theta) = 2n\log\theta + (n-r)m_{(r)}\theta + \log u(\theta) - n\log(1+\theta),$$

and $v(\theta) = \ln b - b\theta$. So, Eq. (3.8) is converted as follows:

$$\hat{U}^h_{Bay}(b) = \left\{ \frac{\int_0^\infty [h(\theta)]^{-k} e^{w(\theta)} d\theta}{\int_0^\infty e^{w(\theta)} d\theta} \right\}^{-\frac{1}{k}}$$
$$= \left( [h(\theta)]^{-k} + \frac{1}{2} h_{11}\delta_{11} + v_1 h_1 \delta_{11} + \frac{1}{2} w_3 (\delta_{11})^2 h_1 \right)^{-\frac{1}{k}}, \qquad (3.10)$$

where

$$h_1 = \frac{d([h(\theta)]^{-k})}{d\theta}, \quad h_{11} = \frac{d^2([h(\theta)]^{-k})}{d\theta^2},$$
$$v_1 = \frac{dv(\theta)}{d\theta}, \quad w_3 = \frac{d^3 w(\theta)}{d^3\theta},$$

and

$$\delta_{11} = \left[ -\frac{d^2 w(\theta)}{d^2\theta} \right]^{-1}.$$

By utilizing Eq. (3.10) and considering the specific case where $h(\theta) = \theta$, we can derive the Bayesian estimate for $\theta$ as follows:

$$\hat{\theta}_{Bay}(b) = \hat{U}^\theta_{Bay}(b)$$
$$= \left( \theta^{-k} + \frac{k(k+1)\delta_{11}}{2} \theta^{-(k+2)} + bk\delta_{11}\theta^{-(k+1)} - \frac{kw_3(\delta_{11})^2}{2} \right)^{-\frac{1}{k}}, \quad (3.11)$$

where $\delta_{11}$ and $w_3$ are

$$\delta_{11} = \left( \frac{2n}{\theta^2} - \frac{n}{(1+\theta)^2} - \sum_{j=1}^{r} \frac{I_2^j I_0^j - (I_1^j)^2}{(I_0^j)^2} \right)^{-1},$$
$$w_3 = \frac{4n}{\theta^3} - \frac{2n}{(1+\theta)^3} + \sum_{j=1}^{r} \frac{I_3^j (I_0^j)^2 - 3 I_0^j I_1^j I_2^j + 2(I_1^j)^3}{(I_0^j)^3}, \qquad (3.12)$$

and

$$I_n^j = \int_0^\infty x^n (1+x) e^{-\theta x} \mu_{\tilde{x}_j}(x) dx, \quad n = 0, 1, 2, 3.$$

**Definition 3.1.** *If the Bayesian estimate of the parameter $\theta$ is denoted as $\hat{\theta}_{Bay}(b)$, and the prior distribution of $b$ is represented by $\pi_1(b)$, then the E-Bayesian estimate of the parameter $\theta$, which is the mathematical expectation of $\hat{\theta}_{Bay}(b)$, can be denoted as $\hat{\theta}_{EBay}$ and defined as follows:*

$$\hat{\theta}_{EBay} = \int_\lambda \hat{\theta}_{Bay}(b) \pi_1(b) db, \quad b \in \Lambda. \qquad (3.13)$$

According to Equations (3.11) and (3.13) and the distribution of $b$, the E-Bayesian estimation of $\alpha$ is obtained as follows:

$$\begin{aligned}\hat{\theta}_{EBay} &= \frac{1}{c}\int_0^c \left(\theta^{-k} + \frac{k(k+1)\delta_{11}}{2}\theta^{-(k+2)} + bk\delta_{11}\theta^{-(k+1)} - \frac{kw_3(\delta_{11})^2}{2}\right)^{-\frac{1}{k}} db \\ &= \frac{1}{c(k-1)\delta_{11}}\left\{\left(\theta^{-k} + \frac{k(k+1)\delta_{11}}{2}\theta^{-(k+2)} + ck\delta_{11}\theta^{-(k+1)} - \frac{kw_3(\delta_{11})^2}{2}\right)^{\frac{k-1}{k}}\right. \\ &\quad \left. - \left(\theta^{-k} + \frac{k(k+1)\delta_{11}}{2}\theta^{-(k+2)} - \frac{kw_3(\delta_{11})^2}{2}\right)^{\frac{k-1}{k}}\right\}, \qquad (3.14)\end{aligned}$$

where $\delta_{11}$ and $w_3$ are given in Eq. (3.12).

## 4. Hierarchical Bayesian estimation

In this section, our focus will be on establishing the prior hierarchical density function and performing hierarchical Bayesian estimation for $\theta$. Assuming $\lambda$ as the hyperparameter of $\theta$, we denote the prior density function of $\theta$ as $\pi(\theta|\lambda)$. Similarly, the prior density function of the hyperparameter $\lambda$ is denoted as $\pi_1(\lambda)$. Subsequently, we define the hierarchical prior density function for $\theta$ as follows:

$$\pi_2(\theta) = \int_\Lambda \pi(\theta|\lambda)\pi_1(\lambda)d\lambda, \quad \lambda \in \Lambda. \qquad (4.15)$$

Based on Eq. (4.15), we can derive the prior density function of $\theta$ as follows:

$$\pi_2(\theta) = \int_0^c \pi(\theta|b)\pi_1(b)db = \frac{1-(1+c\theta)e^{-c\theta}}{c\theta^2}. \qquad (4.16)$$

By utilizing Equations (2.3) and (4.16), we can derive the hierarchical prior density function of $\theta$ in the following manner:

$$\pi^*(\theta|\tilde{\mathbf{x}}) = \frac{\frac{\theta^{2n}}{(1+\theta)^n}e^{-\theta\left[b+(n-r)m_{(r)}\right]}u(\theta)\pi_2(\theta)}{\int_0^\infty \frac{\theta^{2n}}{(1+\theta)^n}e^{-\theta\left[b+(n-r)m_{(r)}\right]}u(\theta)\pi_2(\theta)d\theta}. \qquad (4.17)$$

Using Eq. (4.17), we can obtain the hierarchical Bayesian estimation of $\theta$ by employing the entropy loss function in the following manner:

$$\hat{\theta}_{HBay} = \left\{\frac{\int_0^\infty \frac{\theta^{2n-k}}{(1+\theta)^n}e^{-\theta\left[b+(n-r)m_{(r)}\right]}u(\theta)\pi_2(\theta)d\theta}{\int_0^\infty \frac{\theta^{2n}}{(1+\theta)^n}e^{-\theta\left[b+(n-r)m_{(r)}\right]}u(\theta)\pi_2(\theta)d\theta}\right\}^{-\frac{1}{k}}.$$

It is evident that estimating $\theta$ is generally not possible. As a result, the Lindley approximation method is commonly used for its estimation. By assuming $w^*(\theta) =$

$\ln l(\tilde{\mathbf{x}}, \theta) + \ln \pi_2(\theta) = L(\theta) + v(\theta)$, where

$$L(\theta) = 2n \log \theta + (n-r) m_{(r)} \theta + \log u(\theta) - n \log(1+\theta)$$
$$v^*(\theta) = \ln[1 - (1 + c\theta) e^{-c\theta}] - \ln(c\theta^2).$$

We have,

$$\hat{\theta}_{HBay} = \left( \theta^{-k} + \frac{1}{2} h_{11} \delta_{11}^* + v_1^* h_1 \delta_{11}^* + \frac{1}{2} w_3^* h_1 (\delta_{11}^*)^2 \right)^{-\frac{1}{k}},$$

where

$$h_1 = \frac{d(\theta^{-k})}{d\theta}, \quad h_{11} = \frac{d^2(\theta^{-k})}{d\theta^2},$$

$$v_1^* = \frac{dv^*(\theta)}{d\theta}, \quad w_3^* = \frac{d^3 w^*(\theta)}{d^3\theta},$$

and

$$\delta_{11}^* = \left[ -\frac{d^2 w^*(\theta)}{d^2\theta} \right]^{-1}.$$

Therefore, we can obtain the hierarchical Bayesian estimation of $\theta$ as follows:

$$\hat{\theta}_{HBay} = \left( \theta^{-k} + \frac{k(k+1)\delta_{11}^*}{2} \theta^{-(k+2)} - \frac{k \left[ c^2\theta^2 e^{-c\theta} + 2(1+c\theta)e^{-c\theta} - 2 \right] \delta_{11}^*}{\theta \left[ 1 - (1+c\theta)e^{-c\theta} \right]} \theta^{-(k+1)} \right.$$
$$\left. - \frac{k w_3^* (\delta_{11}^*)^2}{2} \right), \tag{4.18}$$

where $\delta_{11}^*$ and $w_3^*$ are as below:

$$\delta_{11}^* = \left[ \delta_{11}^{-1} - \frac{c^2 e^{-c\theta} \left( 1 - c\theta - e^{-c\theta} \right)}{\left[ 1 - (1+c\theta)e^{-c\theta} \right]^2} - \frac{2}{\theta^2} \right]^{-1},$$

and

$$w_3^* = w_3 + \frac{c^3 e^{-c\theta} \left( 4e^{-c\theta} + c^2\theta e^{-c\theta} - 2e^{-2c\theta} - 2 \right)}{\left[ 1 - (1+c\theta)e^{-c\theta} \right]^3} - \frac{4}{\theta^3},$$

so that $\delta_{11}$ and $w_3$ are shown in Eq. (3.12).

# 5. Numerical analysis: a simulation study

This section will focus on comparing the Bayesian estimation, E-Bayesian estimation, and hierarchical Bayesian estimation of the parameter $\theta$. The steps for conducting the simulation are outlined as follows:

**Step 1**:
$b$ is generated for a specific value of $c$ and using the prior distribution of $\pi_1(b) = \frac{1}{c}$, where $0 < b < c$.

**Step 2**:
In the first step, $\theta$ is calculated using the estimated value of $b$ and Eq. (3.6).

**Step 3**:
In the process following the estimation of $\theta$ from the second phase, we craft a set of Type-II censored samples using a variety of $(n, r)$ pairings within the framework of the Lindley distribution. This distribution is defined by a probability density function (PDF) outlined in Eq. (1.1), and the sample creation is done through a specific procedure. Confronted with the challenge that the equation $F(x) = u$, or $\theta x - \log(1 + \theta + \theta x) + \log[(1 + \theta)(1 - u)] = 0$ cannot be directly solved, which involves the term $u$ drawn from a uniform distribution over the range $(0, 1)$, we cannot employ the direct inversion method to derive random samples from the Lindley distribution. Nevertheless, we can utilize our understanding that the Lindley distribution is actually a unique combination of the $Exponential(\theta)$ and $Gamma(2, \theta)$ distributions. The function $f(x)$ is a weighted sum of two functions, $pf_1(x)$ and $(1-p)f_2(x)$, for positive values of $x$ and $\theta$, where $p$ is defined as $p = \theta/(1+\theta)$, $f_1(x)$ is $\theta e^{-\theta x}$, and $f_2(x)$ is $\theta^2 x e^{-\theta x}$. To simulate random variables $X_i$ for $i$ ranging from 1 to $n$, according to the Lindley distribution parameterized by $\theta$, we adhere to the subsequent algorithm:

- For each $i$ from 1 to $n$, generate $U_i$ following a $Uniform(0, 1)$ distribution.

- For each $i$ from 1 to $n$, generate $V_i$ from an Exponential distribution with parameter $\theta$.

- For each $i$ from 1 to $n$, generate $W_i$ from a Gamma distribution with the shape parameter 2 and scale parameter $\theta$.

Then for each $i$, if $U_i$ is less than $p = \theta/(1 + \theta)$, we assign $X_i$ the value of $V_i$. If not, $X_i$ is assigned the value of $W_i$. Subsequently, in our third step, we assess every Type-II censored sample $X$ using the fuzzy system as introduced by Pak et al. (2013a). This evaluation is conducted by applying the following membership functions to the fuzzy sample.

$$\mu_{\tilde{x}_1}(x) = \begin{cases} 1 & x \leq 0/25 \\ \frac{0/5-x}{0/25} & 0/25 \leq x \leq 0/5 \\ 0 & otherwise \end{cases}, \quad \mu_{\tilde{x}_2}(x) = \begin{cases} \frac{x-0/25}{0/25} & 0/25 \leq x \leq 0/5 \\ \frac{0/75-x}{0/25} & 0/5 \leq x \leq 0/75 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_3}(x) = \begin{cases} \frac{x-0/5}{0/25} & 0/5 \leq x \leq 0/75 \\ \frac{1-x}{0/25} & 0/75 \leq x \leq 1 \\ 0 & otherwise \end{cases}, \quad \mu_{\tilde{x}_4}(x) = \begin{cases} \frac{x-0/75}{0/25} & 0/75 \leq x \leq 1 \\ \frac{1/25-x}{0/25} & 1 \leq x \leq 1/25 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_5}(x) = \begin{cases} \frac{x-1}{0/25} & 1 \leq x \leq 1/25 \\ \frac{1/25-x}{0/25} & 1/25 \leq x \leq 1/5 \\ 0 & otherwise \end{cases}, \quad \mu_{\tilde{x}_6}(x) = \begin{cases} \frac{x-1/25}{0/25} & 1/25 \leq x \leq 1/5 \\ \frac{1/75-x}{0/25} & 1/5 \leq x \leq 1/75 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_7}(x) = \begin{cases} \frac{x-1/5}{0/25} & 1/5 \leq x \leq 1/75 \\ \frac{2-x}{0/25} & 1/75 \leq x \leq 2 \\ 0 & otherwise \end{cases}, \quad \mu_{\tilde{x}_8}(x) = \begin{cases} x - 1/75 & 1/75 \leq x \leq 2 \\ 1 & x \geq 2 \\ 0 & otherwise \end{cases}$$

To calculate the estimate of $\theta$, we utilized Equations (3.11), (3.14), and (4.18) for the Bayesian, E-Bayesian, and hierarchical Bayesian approaches, respectively. This estimation procedure, encompassing the initial three steps, was conducted 5000 times. Subsequently, we computed the mean estimate and the mean squared error for each calculation, which are detailed in Tables 1 through 3.

Table 1: The average value (AV) and mean squared error (MSE) of the estimates of $\theta$ for various combinations of $(n, r)$, with $k = 2.5$ and $c = 2$.

| | | $\hat{\theta}_{\text{Bay}}$ | | $\hat{\theta}_{\text{EB}}$ | | $\hat{\theta}_{\text{HB}}$ | |
|---|---|---|---|---|---|---|---|
| $n$ | $r$ | AV | MSE | AV | MSE | AV | MSE |
| 15 | 10 | 0.45919 | 0.23565 | 2.5596 | 8.0475 | 0.45791 | 0.23496 |
| 15 | 12 | 0.58995 | 0.34152 | 4.1179 | 1.2697 | 0.58905 | 0.34060 |
| 30 | 10 | 0.49601 | 0.26755 | 9.0763 | 2.0329 | 0.49493 | 0.26652 |
| 30 | 15 | 0.75936 | 0.50453 | 2.2164 | 5.7672 | 0.75883 | 0.50402 |
| 30 | 20 | 0.88733 | 0.70995 | 1.6719 | 5.2871 | 0.88702 | 0.70963 |
| 50 | 30 | 1.17700 | 1.02620 | 1.3266 | 4.1612 | 1.17680 | 1.02610 |
| 50 | 40 | 1.28500 | 1.25210 | 1.4989 | 3.4333 | 1.28490 | 1.25200 |
| 100 | 50 | 1.38030 | 1.47880 | 2.3151 | 7.2344 | 1.38020 | 1.47870 |
| 100 | 75 | 1.57846 | 1.92048 | 2.2591 | 5.7384 | 1.54847 | 1.92045 |
| 200 | 120 | 1.72474 | 0.24477 | 3.4616 | 1.0935 | 1.72472 | 0.24477 |
| 200 | 150 | 1.82367 | 0.27103 | 4.6589 | 1.4728 | 1.82366 | 0.27103 |
| 300 | 200 | 1.93404 | 0.31066 | 6.0102 | 1.8993 | 1.93402 | 0.31066 |

Table 2: The average value (AV) and mean squared error (MSE) of the estimates of $\theta$ for various combinations of $(n, r)$, with $k = 1.5$ and $c = 3$.

| | | $\hat{\theta}_{\text{Bay}}$ | | $\hat{\theta}_{\text{EB}}$ | | $\hat{\theta}_{\text{HB}}$ | |
|---|---|---|---|---|---|---|---|
| $n$ | $r$ | AV | MSE | AV | MSE | AV | MSE |
| 15 | 10 | 0.387905 | 0.237101 | 8.84804 | 2.78477 | 0.386140 | 0.235024 |
| 15 | 12 | 0.549023 | 0.391670 | 1.720587 | 5.440366 | 5.476302 | 0.390088 |
| 30 | 10 | 0.456275 | 0.286303 | 1.54725 | 4.89115 | 0.454268 | 0.284306 |
| 30 | 15 | 0.772446 | 0.654157 | 4.25977 | 1.34706 | 0.771869 | 0.653346 |
| 30 | 20 | 0.895936 | 1.011226 | 6.68305 | 1.31643 | 0.895666 | 1.010924 |
| 50 | 30 | 1.113385 | 1.644294 | 1.71068 | 5.39716 | 1.113254 | 1.164217 |
| 50 | 40 | 1.321879 | 0.215920 | 8.29087 | 2.62177 | 1.321870 | 0.215913 |
| 100 | 50 | 1.380530 | 2.807087 | 8.64223 | 2.71223 | 1.380442 | 2.807085 |
| 100 | 75 | 1.718225 | 0.407471 | 8.92795 | 2.82326 | 1.718230 | 0.407470 |
| 200 | 120 | 1.778132 | 0.533849 | 4.50165 | 1.42334 | 1.778122 | 0.533677 |
| 200 | 150 | 1.829669 | 0.527371 | 4.20076 | 1.328283 | 1.829686 | 0.527371 |
| 300 | 200 | 1.860709 | 0.573782 | 1.74237 | 5.45145 | 1.860698 | 0.573781 |

Table 3: The average value (AV) and mean squared error (MSE) of the estimates of $\theta$ for various combinations of $(n, r)$, with $k = 3$ and $c = 1.5$.

| | | $\hat{\theta}_{\text{Bay}}$ | | $\hat{\theta}_{\text{EB}}$ | | $\hat{\theta}_{\text{HB}}$ | |
|---|---|---|---|---|---|---|---|
| $n$ | $r$ | AV | MSE | AV | MSE | AV | MSE |
| 15 | 10 | 0.504677 | 0.224393 | 2.15262 | 6.79651 | 0.503779 | 0.223762 |
| 15 | 12 | 0.647202 | 0.317141 | 1.08086 | 2.70186 | 0.646563 | 0.316616 |
| 30 | 10 | 0.538610 | 0.251183 | 1.46428 | 4.50750 | 0.537780 | 0.250598 |
| 30 | 15 | 0.829147 | 0.458681 | 3.25068 | 1.02778 | 0.829303 | 0.458268 |
| 30 | 20 | 1.014993 | 0.614151 | 6.97603 | 2.20597 | 1.014723 | 0.613925 |
| 50 | 30 | 1.200791 | 0.896351 | 2.81974 | 8.91677 | 1.200691 | 0.896255 |
| 50 | 40 | 1.330317 | 0.107214 | 4.13114 | 1.12505 | 1.330238 | 0.107205 |
| 100 | 50 | 1.511448 | 0.127547 | 5.10061 | 1.57374 | 1.511366 | 0.127542 |
| 100 | 75 | 1.669456 | 0.160740 | 3.89875 | 1.23289 | 1.669419 | 0.160737 |
| 200 | 120 | 1.936836 | 0.207105 | 9.82617 | 2.61061 | 1.936823 | 0.207103 |
| 200 | 150 | 2.102049 | 0.236256 | 2.95966 | 7.47460 | 2.102034 | 0.236255 |
| 300 | 200 | 2.151874 | 0.258583 | 3.15324 | 7.14100 | 2.151865 | 0.258582 |

## 6. Real dataset analysis

In this section, we present a real dataset to demonstrate that the Lindley distribution may provide a superior model compared to the exponential distribution. The dataset in Table 4 represents the waiting times (in minutes) before service for 100 bank customers (Ghitany et al. (2008)). Using real data, the Kolmogorov–Smirnov test statistic and the corresponding p-value were obtained as 0.1126 and 0.8765, respectively, using the R software. These results indicate that the data fit the Lindley distribution well.

From the uniform distribution in the interval $[0, 0.5]$, we randomly select a number and call it $u_1$, and from the uniform distribution in the interval $[0.5, 1]$, we randomly select another number and call it $u_2$. Then, we convert the real data $(X)$ into triangular fuzzy data in the form $(X, X \cdot u_1, X \cdot u_2)$ (see Table 5).

The Bayesian estimates, E-Bayesian, and hierarchical Bayesian estimate of the Lindley parameter, along with the Kolmogorov–Smirnov (K-S) test statistic and the corresponding p-value, are presented in Table 6. Given the values of the K-S test statistic and its p-value, it is clear that the hierarchical Bayesian estimate of the Lindley parameter is better than the Bayesian and E-Bayesian estimates.

Table 4: Waiting Times (min) of 100 Bank Customers

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0.8 | 0.8 | 1.3 | 1.5 | 1.8 | 1.9 | 1.9 | 2.1 | 2.6 | 2.7 |
| 2.9 | 3.1 | 3.2 | 3.3 | 3.5 | 3.6 | 4.0 | 4.1 | 4.2 | 4.2 |
| 4.3 | 4.3 | 4.4 | 4.4 | 4.6 | 4.7 | 4.7 | 4.8 | 4.9 | 4.9 |
| 5.0 | 5.3 | 5.5 | 5.7 | 5.7 | 6.1 | 6.2 | 6.2 | 6.2 | 6.3 |
| 6.7 | 6.9 | 7.1 | 7.1 | 7.1 | 7.1 | 7.4 | 7.6 | 7.7 | 8.0 |
| 8.2 | 8.6 | 8.6 | 8.6 | 8.8 | 8.8 | 8.9 | 8.9 | 9.5 | 9.6 |
| 9.7 | 9.8 | 10.7 | 10.9 | 11.0 | 11.0 | 11.1 | 11.2 | 11.2 | 11.5 |
| 11.9 | 12.4 | 12.5 | 12.9 | 13.0 | 13.1 | 13.3 | 13.6 | 13.7 | 13.9 |
| 14.1 | 15.4 | 15.4 | 17.3 | 17.3 | 18.1 | 18.2 | 18.4 | 18.9 | 19.0 |
| 19.9 | 20.6 | 21.3 | 21.4 | 21.9 | 23.0 | 27.0 | 31.6 | 33.1 | 38.5 |

Table 5: Fuzzified data represented as triangular fuzzy numbers $(X, X.u_1, X.u_2)$

| $(X, X.u_1, X.u_2)$ | $(X, X.u_1, X.u_2)$ | $(X, X.u_1, X.u_2)$ | $(X, X.u_1, X.u_2)$ | $(X, X.u_1, X.u_2)$ |
|---|---|---|---|---|
| (0.8, 0.405, 0.205) | (0.8, 0.029, 0.607) | (1.3, 0.035, 1.06) | (1.5, 0.582, 0.656) | (1.8, 0.186, 1.55) |
| (2.9, 2.75, 2.29) | (3.1, 0.041, 0.925) | (3.2, 3.05, 0.473) | (3.3, 1.54, 2.80) | (3.5, 2.61, 0.151) |
| (4.3, 0.202, 4.12) | (4.3, 3.39, 3.09) | (4.4, 0.049, 0.814) | (4.4, 1.38, 0.090) | (4.6, 2.34, 0.300) |
| (5.0, 3.36, 4.05) | (5.3, 4.97, 3.95) | (5.5, 0.713, 5.13) | (5.7, 1.19, 2.49) | (5.7, 3.86, 1.49) |
| (6.7, 3.71, 4.40) | (6.9, 1.95, 4.75) | (7.1, 4.63, 3.85) | (7.1, 5.32, 6.56) | (7.1, 0.959, 0.914) |
| (8.2, 7.11, 3.33) | (8.6, 1.06, 2.51) | (8.6, 0.534, 0.611) | (8.6, 7.17, 1.04) | (8.8, 4.47, 8.29) |
| (9.7, 5.49, 4.93) | (9.8, 9.47, 4.93) | (10.7, 0.352, 5.39) | (10.9, 4.08, 1.28) | (11.0, 9.12, 6.58) |
| (11.9, 3.48, 10.5) | (12.4, 4.30, 6.27) | (12.5, 8.29, 9.61) | (12.9, 5.60, 11.3) | (13.0, 5.63, 8.70) |
| (14.1, 2.68, 11.6) | (15.4, 12.6, 11.9) | (15.4, 0.587, 13.4) | (17.3, 0.368, 10.2) | (17.3, 10.9, 0.129) |
| (19.9, 2.59, 12.9) | (20.6, 15.8, 12.1) | (21.4, 1.98, 15.5) | (21.4, 11.9, 8.70) | (21.9, 10.1, 4.28) |
| (1.9, 1.73, 0.688) | (1.9, 0.648, 1.04) | (2.1, 1.92, 0.450) | (2.6, 0.209, 0.549) | (2.7, 0.694, 2.02) |
| (3.6, 1.09, 3.18) | (4.0, 3.31, 1.39) | (4.1, 1.87, 0.790) | (4.1, 3.95, 3.82) | (4.2, 2.98, 2.81) |
| (4.7, 2.94, 0.148) | (4.7, 0.998, 0.456) | (4.8, 4.53, 2.74) | (4.8, 1.57, 2.55) | (4.9, 1.37, 2.45) |
| (6.1, 3.52, 2.02) | (6.2, 3.36, 4.25) | (6.2, 3.78, 0.299) | (6.2, 2.13, 1.21) | (6.3, 4.42, 0.899) |
| (7.1, 1.39, 4.82) | (7.4, 4.65, 2.60) | (7.6, 6.07, 5.96) | (7.7, 0.399, 3.12) | (8.0, 5.44, 0.747) |
| (8.8, 0.275, 6.73) | (8.9, 4.32, 0.731) | (8.9, 6.89, 3.42) | (9.5, 6.82, 4.38) | (9.6, 7.38, 2.09) |
| (11.0, 2.53, 4.54) | (11.1, 4.04, 2.05) | (11.2, 9.65, 6.75) | (11.2, 11.0, 2.59) | (11.5, 6.41, 7.07) |
| (13.1, 2.82, 9.09) | (13.3, 7.51, 7.00) | (13.6, 11.6, 3.39) | (13.7, 8.87, 1.69) | (13.9, 4.42, 0.244) |
| (18.1, 11.6, 11.3) | (18.2, 18.1, 2.14) | (18.4, 8.35, 17.8) | (18.9, 11.6, 17.5) | (19.0, 0.209, 11.2) |
| (23.0, 11.7, 5.29) | (27.0, 17.2, 5.74) | (31.6, 13.3, 4.58) | (33.1, 13.1, 9.05) | (38.5, 14.7, 31.7) |

Table 6: Kolmogorov–Smirnov test statistics and Lindley parameter estimates

| **Metric** | $\theta_{\text{HB}}$ | $\theta_{\text{EB}}$ | $\theta_{\text{Bay}}$ |
|---|---|---|---|
| Estimate | 0.5046 | 2.152 | 0.5037 |
| K-S statistic | 0.0329 | 0.0594 | 0.0476 |
| P-value | 0.9261 | 0.7234 | 0.8995 |

# 7. Conclusion and Future Work

This study provides practical insights into parameter estimation for reliability data where measurements are imprecise and censored. The proposed Bayesian framework, incorporating fuzzy data and flexible prior distributions, offers a robust and interpretable approach suitable for real-world decision-making. Applications span a wide range of fields, including industrial reliability, service optimization, and medical diagnostics, where uncertainty is an inherent challenge. By highlighting the advantages of Bayesian inference and the use of the Lindley model, this work contributes a versatile methodology with both theoretical depth and practical relevance.

In our research, we estimated the parameter of the Lindley distribution under a Type-II censoring scheme, using fuzzy data and an entropy loss function for evaluation. Our analysis involved comparing Bayesian estimation, E-Bayesian estimation, and hierarchical Bayesian estimation, all within the framework of Monte Carlo simulations. The results indicated that hierarchical Bayesian estimation outperformed both Bayesian and E-Bayesian methods in terms of efficiency. Moreover, the Bayesian approach demonstrated better performance than the E-Bayesian method. Notably, as the sample size increased, the performance of the hierarchical and standard Bayesian estimators tended to converge, ultimately showing negligible differences.

Suggestions for future research are as follows:

- Extending the methodology to other lifetime distributions beyond the Lindley model;
- Investigating the impact of different types of fuzzy sets (e.g., intuitionistic or interval-valued fuzzy sets);
- Applying advanced MCMC algorithms, such as Hamiltonian Monte Carlo, to improve sampling efficiency and convergence;
- Conducting a comparative analysis with frequentist estimators, such as Maximum Likelihood Estimation (MLE), to evaluate trade-offs between Bayesian and classical approaches under uncertainty.

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