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Research paper

Solving The Black-Scholes Problem Using a Combined Numerical Method (A Case Study of Tehran Stock Exchange)

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Abstract:
The Black-Scholes model is one of the most widely used frameworks for pricing options in financial markets. However, its analytical solutions are often limited to idealized conditions, necessitating the use of numerical methods for more complex scenarios. This study proposes a combined numerical approach to solve the Black-Scholes equation, specifically focusing on call option pricing in the context of Iran's financial market. The proposed method integrates fully implicit and explicit methods to enhance accuracy and computational efficiency. By applying this approach to historical data from the Iranian options market, we demonstrate its effectiveness in capturing market dynamics and pricing call options under local conditions. The results indicate that the combined numerical method not only provides reliable pricing estimates but also offers insights into the unique characteristics of option trading in emerging markets like Iran. This research contributes to the growing body of literature on numerical methods in financial engineering and provides practical tools for traders and analysts in developing economies.

Keywords: Black-Scholes, Numerical methods, Option trading, Iran market.
Classification: 91Gxx, 49Mxx.

1 Introduction

The Black-Scholes equation, introduced by Fischer Black and Myron Scholes in 1973, revolutionized the field of financial mathematics by providing a theoretical framework for pricing European options [1]. The model is formulated as a partial

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differential equation (PDE) that describes the evolution of option prices over time, considering factors such as underlying asset price, volatility, risk-free interest rate, and time to maturity. Despite its widespread adoption, the Black-Scholes model relies on several simplifying assumptions, such as constant volatility and the absence of transaction costs, which limit its applicability to real-world markets [2].

In practice, analytical solutions to the Black-Scholes PDE are only available for simple scenarios, such as European call and put options. For more complex financial instruments or market conditions, numerical methods are often employed to approximate solutions. Common numerical approaches include finite difference methods (FDM), Monte Carlo simulations, and finite element methods (FEM) [3]. Each method has its strengths and limitations, and the choice of technique depends on the specific requirements of accuracy, computational efficiency, and stability.

This study focuses on solving the Black-Scholes PDE using a combined numerical approach, with a particular emphasis on pricing call options in the Iranian financial market. The Iranian market, as an emerging economy, presents unique challenges, such as high volatility and regulatory constraints, which necessitate tailored pricing models. By integrating implicit and explicit methods, we aim to develop a robust and efficient method for option pricing under these conditions. The proposed approach not only addresses the limitations of traditional analytical solutions but also provides insights into the dynamics of option trading in emerging markets.

Partial Differential Equations (PDEs) are fundamental tools in modeling a wide range of phenomena in physics, engineering, and finance. These equations describe how quantities change with respect to multiple variables, such as time and space, and are particularly useful for capturing dynamic systems [4–7]. Research on solving PDEs has predominantly relied on analytical and numerical methods [8–11]. However, analytical solutions are often challenging to obtain for certain equations, and conventional numerical approaches tend to involve high computational expenses and lengthy processing times [11]. As a result, the development of hybrid numerical techniques can offer significant efficiency, reducing both computational costs and time.

Numerical methods have been extensively studied for solving the Black-Scholes PDE. Finite difference methods (FDM) are among the most popular techniques due to their simplicity and efficiency. For instance, Wilmott et. al [3] demonstrated the application of explicit and implicit FDM schemes for option pricing, highlighting their stability and convergence properties. Similarly, Tavella and Randall [12] discussed the use of Crank-Nicolson schemes, which combine the stability of implicit methods with the accuracy of explicit methods.

Monte Carlo simulations, another widely used approach, are particularly effective for pricing path-dependent options. Boyle [13] pioneered the use of Monte Carlo methods in finance, showing how random sampling can approximate complex financial derivatives. More recently, Glasserman [14] advanced the field by introducing variance reduction techniques to improve computational efficiency.

Hybrid methods, which combine multiple numerical techniques, have also gained attention. For example, Zhang [15] proposed a combination of FDM and Monte Carlo simulations to address the limitations of each method. These approaches have been shown to enhance accuracy and computational performance, especially for exotic options and complex market conditions.

Graccinti [16] used the Monte Carlo simulation method to calculate the Greeks. Additionally, within the framework of a specific algorithm and variance reduction techniques, they employed the finite difference method to estimate risk sensitivity parameters and the value of options. Andersen [17] introduced a finite difference method for computing sensitivity parameters applicable to a broad class of continuous-time Markov chain models. Moruy and Souda [18] proposed an efficient algorithm for calculating the value of European and American options as well as the risk sensitivity parameters of these options. They also demonstrated that the risk sensitivity parameters for European options are asymptotically equal to the risk sensitivity parameters of Malliavin. Jeong et al. [19] presented a finite difference method for solving the Black-Scholes partial differential equation and pricing options without the need for boundary conditions. In recent years, researchers have continued to refine numerical methods for solving the Black-Scholes PDE, particularly for complex financial instruments and market conditions. For example, Bertram and Fournié [20] proposed a high-order finite difference scheme combined with adaptive mesh refinement to improve the accuracy and efficiency of option pricing under stochastic volatility models. Their results demonstrated significant improvements in computational performance compared to traditional methods. Emerging markets, such as Iran, present unique challenges for option pricing due to factors like high volatility, regulatory constraints, and limited liquidity. Despite these challenges, few studies have focused on adapting numerical methods to these markets. For instance, Golbabai et al. [21] explored the pricing of options Stock Exchange using fractional Black-Scholes models, highlighting the need for localized approaches. Similarly, Jeong et al. [22] developed a hybrid numerical approach integrating finite difference and Monte Carlo simulations to price options. Their method addressed the challenges of early exercise features and achieved robust convergence rates. This study highlights the potential of combining multiple numerical techniques to handle complex boundary conditions and non-linearities in the Black-Scholes framework. Recent studies have also begun to explore the application of numerical methods in emerging markets, where market inefficiencies and regulatory constraints pose unique challenges. For instance, Peymany et al. [23] investigated the pricing of options in the Tehran Stock Exchange using a Black-Scholes model that incorporates local volatility surfaces. Their findings emphasized the importance of calibrating models to reflect the specific characteristics of emerging markets. Also, Peymany [24] presents a comprehensive mathematical modeling of stock price dynamics and option valuation. Furthermore, approximation schemes under more sophisticated models, such as those proposed by Safdari-Vaighani et al. [25], offer robust tools for option pricing in environments with stochastic parameters. Numerical approaches based on radial basis function partition of unity methods, as discussed in [26], have also proved valuable in solving convectiondiffusion equations arising in financial contexts.

While significant progress has been made in developing numerical methods for option pricing, several gaps remain. First, most studies focus on developed markets, with limited attention to emerging economies like Iran. Second, the application of hybrid numerical methods in these markets is underexplored. Finally, there is a need for empirical studies that validate the effectiveness of these methods using real-world data from emerging markets.

This study addresses these gaps by proposing a combined numerical approach for solving the Black-Scholes PDE, with a focus on call option pricing in the Iranian market. By integrating implicit and explicit, we aim to provide a robust and efficient framework for option pricing in emerging economies. The primary novelty and main contributions of this paper are summarized and highlighted below. While the Black-Scholes equation and its applications in option pricing have been widely investigated, this study introduces methodological advances and practical implementations that distinguish it from previous works.

Application of a Combined Numerical Method to the Black-Scholes Equation:

While the Black-Scholes equation has been extensively studied, our approach utilizes a hybrid numerical technique that combines the strengths of finite difference methods with iterative schemes to improve stability and accuracy. This specific formulation and implementation, particularly tailored for option pricing problems, is rarely addressed in existing literature, especially in the context of real trading conditions.

Adaptation to the Tehran Stock Exchange (TSE):

One of the most significant contributions of this study is the customization and application of the Black-Scholes framework to the Iranian market. To our knowledge, very limited research has implemented or calibrated the Black-Scholes model using real option contracts from the TSE. The trading structure, liquidity, and regulatory mechanisms in TSE are fundamentally different from Western markets, and this adaptation addresses a real and practical gap in the literature.

Validation Using Real Market Data:

Our study performs a direct comparison between the computed results and actual option prices observed in the TSE. This level of empirical validation using local financial instruments provides actionable insights for both researchers and practitioners interested in emerging markets and underrepresented financial environments.

Practical Recommendations for Local Market Use:

The results and error analysis in our study provide practical guidelines for brokers, investors, and financial regulators in Iran regarding how numerical techniques can be deployed in real-world option valuation something that is largely missing in previous theoretical treatments.

Bridging Mathematical Modeling and Market Implementation:
We have made efforts to bridge the gap between pure mathematical modeling and its computational implementation for actual trading scenarios, specifically call options trading under Iranian financial context, which has not been comprehensively addressed in earlier works.

Algebraic System Solved Over Full Spatiotemporal Grid:
Another important contribution of this study lies in the complete numerical solution of the algebraic system resulting from the discretization of the Black-Scholes partial differential equation over the entire spatiotemporal domain. Unlike many prior studies that limit the solution to a fixed strike price or a snapshot in time, we solve the system at all time steps and asset price levels. This approach enables a more comprehensive understanding of the option's value evolution over time. This full-grid solution allows us to:

- Accurately estimate option prices not only at maturity but at any point in time prior to expiration.

- Analyze how changes in market variables (such as volatility or underlying asset price) affect the option value dynamically.

- Support scenario-based forecasting and sensitivity analysis for traders and analysts, which is especially valuable in markets like Iran where market data may be limited or volatile.

- Provide a flexible computational framework that can be adapted for early-exercise decisions in American options, future research on path-dependent derivatives, or stochastic volatility models.

- Therefore, this comprehensive solution framework significantly enhances the practical utility of the model and improves its relevance for real-world decision-making in financial markets with specific expiration dates and trading constraints.

The remainder of this paper is organized as follows: Section 2 reviews the mathematical formulation of the Black-Scholes PDE and its boundary conditions and also describes the proposed numerical methods and their implementation. Section 3 presents the results of applying the method to historical data from the Iranian options market, and Section 4 discusses the implications of the findings for both academic research and practical applications.

2 Numerical methods

In this section, we will examine the Black-Scholes problem, the form of partial differential equations, and its discretization to solve this equation using available data. The Black-Scholes model plays a fundamental and central role in pricing and hedging options in financial engineering. The core idea of the Black-Scholes model is to determine how stock price fluctuations will behave over time. The primary assumption of the model is that stock prices follow a random walk, and short-term price changes follow a log-normal distribution. In the Black-Scholes model, the price of the underlying asset (stock) over a short time period follows a geometric Brownian motion. Brownian motion has no memory, meaning it forgets its past, which is why the Black-Scholes model aligns with the behavior of ideal financial markets. The Black-Scholes partial differential equation is expressed as follows:

The Black-Scholes partial differential equation is given by:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{1}$$

where: $V(S,t)$ is Option price at time t with underlying asset price S, S is Underlying asset (stock) price, t, Time remaining until the option's expiration, σ is Volatility (standard deviation of the asset's returns) and r is Risk-free interest rate. **Boundary conditions** for a European call option are as follows:

- At maturity $t = T$:

  $$V(S,T) = \max(S-K,0)$$

  where K is the strike price of the option.

- For $S = 0$ (i.e., when the stock price reaches zero):

  $$V(0,t) = 0$$

  This reflects that the value of the option becomes zero if the stock price reaches zero before expiration.

- For $S \to \infty$ (i.e., when the stock price becomes infinitely large):

  $$V(S,t) \approx S - Ke^{-r(T-t)}$$

  As the stock price increases indefinitely, the value of the call option approaches the intrinsic value of the stock minus the discounted strike price.

- For $t \to T$ (at expiration):

  $$V(S,T) = \max(S-K,0) \quad \text{(European Call Option)}$$

  This condition ensures that at the expiration of the option, the option's payoff is the intrinsic value.

- At $S \to \infty$ (at very high stock prices):

$$V(S,t) \to S - Ke^{-r(T-t)}$$

This represents the case when the option becomes deep in the money, and its value approaches the discounted strike price.

Given the above equation and the specified boundary conditions, we discretize the equations using the fully implicit method to obtain the resulting algebraic system.

$$\begin{aligned} V_{i,j+1} - V_i, j =& rV_{i,j} - AiV_{i+1,j+1} + AiV_{i,j+1} - Bi^2V_{i-1,j+1} + 2Bi^2V_{i,j+1} \\ & - Bi^2V_{i+1,j+1}, \quad j \geq 0, \quad i = 1, 2, \cdots, N-1, \end{aligned} \tag{2}$$

where $A = \frac{r}{h}$, $B = \frac{\sigma^2}{2h^2}$, $t = jk$, $S = ih$ and k,h are time and price steps. Using the equations of (1) and (2), the following linear algebraic system of equations are obtained:

$$\mathbf{A}\mathbf{u}^{j+1} + \mathbf{B} = (1+r)\mathbf{u}^j, \tag{3}$$

where $\mathbf{A}$ is a tridiagonal matrix with the following structure:

$$\mathbf{A} = \begin{pmatrix} d_1 & e_1 & 0 & \cdots & 0 \\ f_2 & d_2 & e_2 & \cdots & 0 \\ 0 & f_3 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & e_{n-2} \\ 0 & 0 & 0 & f_{n-1} & d_{n-1} \end{pmatrix}$$

with

$$d_i = 1 - iA - 2(i^2)B, \qquad e_i = iA + (i^2)B, \qquad f_i = i^2 B,$$

and

$$\begin{aligned} &\mathbf{u}^{j+1} = (u_{1,j+1}, u_{2,j+1}, \ldots, u_{N-1,j+1})^T, \\ &\mathbf{B} = (BP_{j+1}, 0, 0, \ldots, 0, 0, (n-1)A + (n-1)^2 BQ_(j+1))^T, \\ &\mathbf{u}^j = (u_{1,j}, u_{2,j}, \ldots, u_{N-1,j})^T. \end{aligned}$$

Discretization: In this discretization of the Black-Scholes equation, the primary objective is to numerically approximate the solution to the partial differential equation (PDE) by transforming it into a system of linear equations. This is often done using finite difference methods (FDM), which discretize the time and asset price variables into small steps and replace the derivatives in the PDE with finite difference approximations.

The discretization shown constructs a system of equations based on the fully implicit method. The advantage of this method is that it is unconditionally stable and second-order accurate in both time and space.

In the matrix provided, the matrix represents the coefficients corresponding to the finite difference approximation of the Black-Scholes equation. Here's a step-by-step explanation of the components involved:

Discretization of the Stock Price and Time: The domain for the stock price S is divided into N discrete points (spatial grid points), and the time domain is divided into $j$time steps. The values $u_{i,j}$ represent the numerical solution at the $i-th$ stock price and the $j-th$ time step.

The matrix on the left-hand side represents the finite difference coefficients. The coefficients A and B represent the discretization terms derived from the Black-Scholes equation.

A is associated with the first-order derivative of the option price with respect to the stock price.

B corresponds to the second-order derivative of the option price with respect to the stock price, which models the diffusion term (volatility). The diagonal terms represent the combination of the finite difference terms from the current and next time steps, while the off-diagonal terms represent the coupling between neighboring grid points. As you move down the matrix, the terms grow progressively larger because of the increase in stock price values (captured by terms like $9B$,$(n-2)^2B$ etc.), reflecting the scaling effect of the second derivative.

Boundary Conditions: The vector on the right-hand side of the equation ensures that boundary conditions are respected. The boundary conditions at the two ends (e.g., $u_{1,j+1}$ and $u_{N-1,j+1}$) ensure that at zero stock price or very high stock prices, the value of the option is handled correctly, as defined by the European call option's payoff function. In the equation for the boundary condition, terms like P_j+1 and Q_j+1 represent contributions from the payoff function or boundary corrections needed to solve the system of equations.

Solving the System: The matrix equation on the left-hand side must be solved at each time step j to find the solution vector $u_i, j+1$, which gives the option price at the next time step. This system is typically solved using numerical linear algebra techniques such as LU decomposition, Gauss-Seidel, or iterative solvers.

Interpretation: The term $(1+r)$ on the right-hand side accounts for the risk-free rate r, which impacts the time decay of the option's value. By iterating over time steps, starting from the terminal condition (the payoff of the option at maturity), the option price at earlier times can be calculated, leading to the price of the option at the initial time. This matrix approach effectively captures the discrete nature of the numerical solution to the Black-Scholes PDE, and the result is a highly efficient way to calculate option prices, particularly for European-style options. The fully-implicit scheme, being stable and accurate, is especially well-suited for this type of problem.

3 Results

In order to demonstrate the effectiveness of the aforementioned method and also to examine it, we examine three examples from the Tehran Stock Exchange market. For numerical computations and simulation results presented in the manuscript, MATLAB R2022 was used on a system equipped with an Intel Core i7-8700K CPU at 3.7 GHz.

3.1 Example 1. The National Iranian Copper Industries Company (NICICO)

The National Iranian Copper Industries Company (NICICO), commonly known as Foolad Melli Iran, is one of the largest copper producers in the Middle East. It plays a critical role in the mining and industrial sectors of Iran, exporting copper to various parts of the world. The company's stock, listed on the Tehran Stock Exchange (TSE) under the ticker FML, is among the most actively traded shares in Iran, reflecting its importance in the national economy. Over the past years, NICICO's stock has demonstrated significant volatility due to factors such as changes in copper prices, global market conditions, and local economic policies. Key aspects influencing its stock performance include: Copper Prices: The price of copper on global markets has a direct impact on NICICO's profitability. Fluctuations in copper demand, supply chain disruptions, or changes in global mining policies all influence the stock price of NICICO. Economic Sanctions: Irans economic sanctions have affected exports, causing some uncertainty in the stock performance. Domestic Policies: Governmental policies related to mining and industrial production, as well as infrastructure development projects, affect NICICO's future prospects.

Options trading (or Options contracts) in the Iranian stock market is relatively new but has been gaining traction as investors look for alternative ways to manage risks and leverage opportunities. The Tehran Stock Exchange has introduced options trading in recent years as part of its financial instruments expansion. Options for NICICO shares typically come with a variety of strike prices and expiration dates. Investors choose the ones that best align with their market expectations. Benefits of Options Trading for NICICO Shares includes the following.
Risk Management: Investors can use options to mitigate risks associated with price volatility in copper or uncertainties due to political and economic factors.
Leverage: Options allow traders to gain exposure to NICICO stock movements with less capital investment compared to purchasing shares outright.
Strategic Flexibility: With options, investors can implement strategies such as straddles, strangles, or covered calls to profit from various market conditions.
NICICOs stock plays a pivotal role in Irans mining industry, and its performance is closely tied to global copper prices and domestic economic factors. As options trading becomes more common in Iran, traders and investors can use these financial instruments to hedge risks and speculate on price movements. Although still

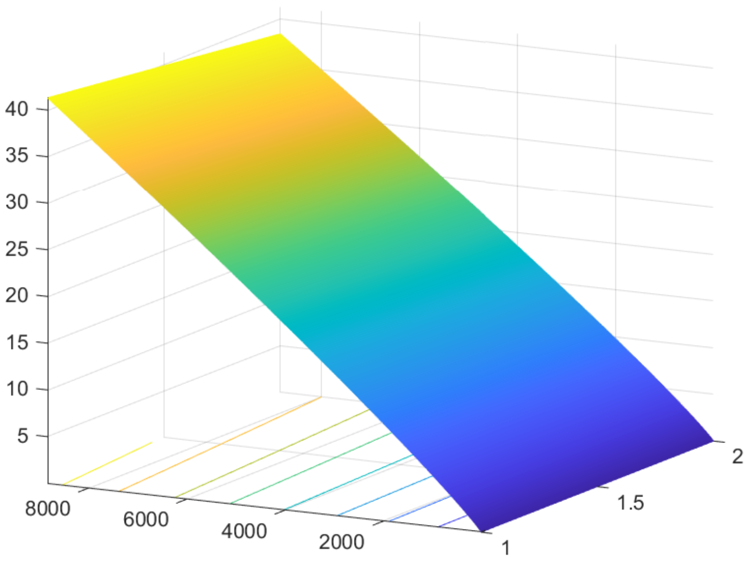
developing, the options market for NICICO provides additional tools for managing investment strategies in Irans evolving financial landscape.
In order to examine the call option contracts for NICICO stock, this example studies the securities labeled with code zamelli3043 (NICICO Option-5000-1404/3/7). The annual volatility of this stock is 1.77586, with a sigma value of 0.27796, the stock price is 8840 IRR (Iranian Rial), the strike price is 5000 IRR, and the time period is considered as 0.2 of the total period. The interest rate in Iran for the year 1404 (2025) was announced at 30%. To calculate the call value for this stock, the algebraic system derived from the Black-Scholes model, with a time step of 0.1 and a price step of 0.01, was solved. Figure 1a presents the solution of this system for all time and price intervals in this example. Figure 1b shows these values for t=0.1 and t=0.2, and Table 1 presents the prices obtained at 20 points corresponding to the figure. As can be seen in the table and in Figure 1b, the calculated call price is 4130.7 IRR, which shows a negligible difference from the exact value obtained from solving the Black-Scholes equation, which is 4131.1 IRR.

Table 1: Numerical results for NICICO

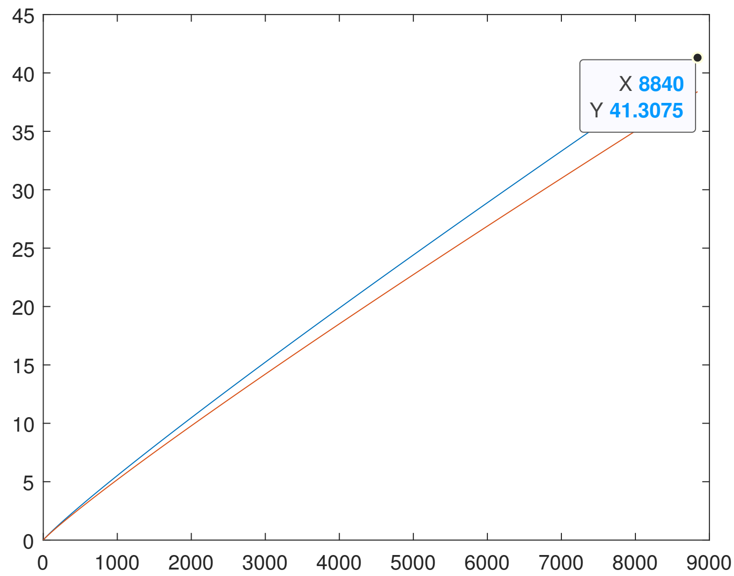
time	$V(S_i) \times 10^2$						
$t = 0.1$	0.009	2.564	4.845	7.033	9.161	11.251	13.303
	15.327	17.328	19.309	21.277	23.225	25.158	27.079
	28.988	30.891	32.779	34.658	36.529	38.396	
$t = 0.2$	0.009	2.731	5.174	7.520	9.805	12.051	14.258
	16.436	18.591	20.724	22.844	24.944	27.028	29.100
	31.159	33.211	35.248	37.275	39.294	**41.307**	

3.2 Example 2. Iran Khodro (IKCO)

Iran Khodro (IKCO) is the largest automotive manufacturer in Iran and one of the largest in the Middle East. Founded in 1962, Iran Khodro produces a wide range of vehicles, including passenger cars, trucks, and buses. The companys stock is listed on the Tehran Stock Exchange (TSE) under the ticker IKCO. Iran Khodro has a significant role in the Iranian economy, and its stock is actively traded, reflecting investor interest in the automotive sector, which is a key industry in Iran.
The performance of Iran Khodros stock is influenced by several factors, both domestically and internationally:
Demand for Vehicles: As Iran's largest car manufacturer, demand for its vehicles, particularly in the domestic market, has a significant impact on stock performance. Economic conditions, consumer purchasing power, and government policies regarding the automotive industry all play roles in shaping this demand. Exchange Rates: Since Iran Khodro relies on some imported components for its vehicles, exchange



(a) The solution $(V(S,t))$.



(b) The solution of NICICO $(V(S))$ in $t=0.1$ and $t=0.2$.

Figure 1: Numerical solution of NICICO.

rate fluctuations (especially the value of the Iranian rial against other currencies) directly affect production costs and profitability, which in turn impact the stock price. Sanctions and International Trade: Sanctions have limited Iran Khodros ability to access foreign markets, affecting both sales and the ability to form partnerships with international automotive companies. Changes in sanctions and international trade relations can cause volatility in IKCOs stock price. Government Support: The automotive industry in Iran benefits from substantial government support, including subsidies and import restrictions on foreign cars, which help maintain Iran Khodros dominant position in the market. Investors holding Iran Khodro shares may use put options to protect their portfolios from downside risk, especially in times of market uncertainty or potential declines in the automotive industry. Covered call strategies are also used by investors who own the stock to generate additional income from the sale of call options while maintaining ownership of the shares.

Benefits of Options Trading for Iran Khodro Shares includes the following.
Risk Mitigation: By using put options, Iran Khodro shareholders can hedge against potential declines in stock price, protecting their portfolios in volatile market conditions.
Income Generation: Selling covered calls on Iran Khodro shares allows investors to generate income through option premiums while maintaining ownership of the stock.
Leverage for Higher Returns: Options allow traders to take advantage of stock price movements with less capital, which can lead to higher percentage returns compared to buying the stock outright.
Strategic Flexibility: Investors can employ various options strategies, such as straddles, spreads, or protective puts, to profit in different market scenarios and manage their risk exposure more effectively.
Iran Khodro (IKCO) is a cornerstone of the Iranian automotive industry and an important player on the Tehran Stock Exchange. The stocks performance is shaped by domestic vehicle demand, international sanctions, currency exchange rates, and government support. As options trading continues to expand in Iran, it provides investors with valuable tools for managing risk and enhancing returns. While the options market is still in its early stages, the growing use of options for stocks like Iran Khodro demonstrates a move towards more sophisticated financial instruments in Irans capital markets.

In order to examine the call option contracts for Iran Khodro's stock, this example studies the securities labeled with code zakhud3089 (IKCO Option-2400-1404/3/7). The annual volatility of this stock is 2.85612, with a sigma value of 0.32645, the stock price is 3863 IRR (Iranian Rial), the strike price is 2400 IRR, and the time period is considered as 0.2 of the total period. The interest rate in Iran for the year 1404 (2025) was announced at 30%. To calculate the call value for this stock, the algebraic system derived from the Black-Scholes model, with a time step of 0.1 and a price step of 0.1, was solved. Figure 2a presents the solution of this system for all time and price intervals in this example. Figure 2b shows these values for t=0.1 and t=0.2, and Table 2, presents the prices obtained at 20 points corresponding to the figure. As can be seen in the table and in Figure 2b, the calculated call price is 1603.4 IRR, which shows a negligible difference from the exact value obtained from solving the Black-Scholes equation, which is 1603.7 IRR.

Table 2: Numerical results for IKCO

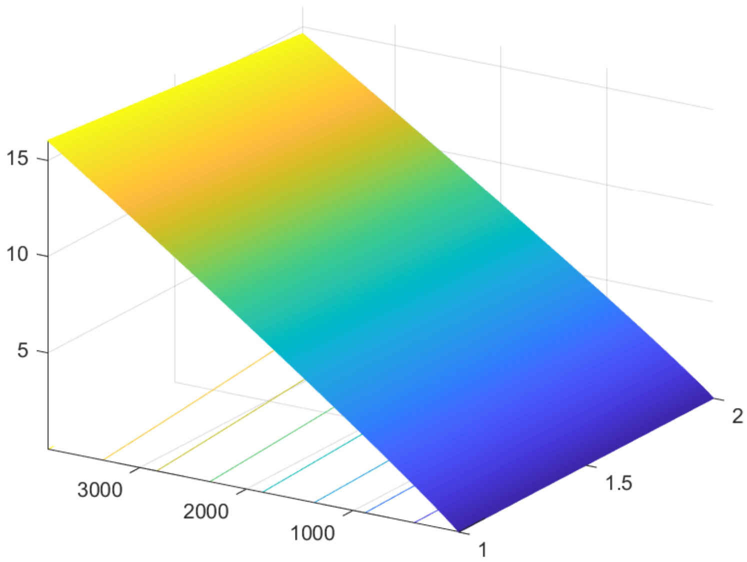
time	$V(S_i) \times 10^2$						
$t = 0.1$	0.006	0.917	1.757	2.573	3.376	4.163	4.942
	5.716	6.481	7.240	7.997	8.747	9.493	10.238
	10.977	11.713	12.449	13.179	13.907	14.636	
$t = 0.2$	0.006	0.998	1.915	2.806	3.685	4.547	5.399
	6.248	7.086	7.918	8.748	9.570	10.388	11.207
	12.017	12.825	13.633	14.434	15.233	**16.034**	

3.3 Example 3. Esfahan's Steel Company (Zob Ahan)

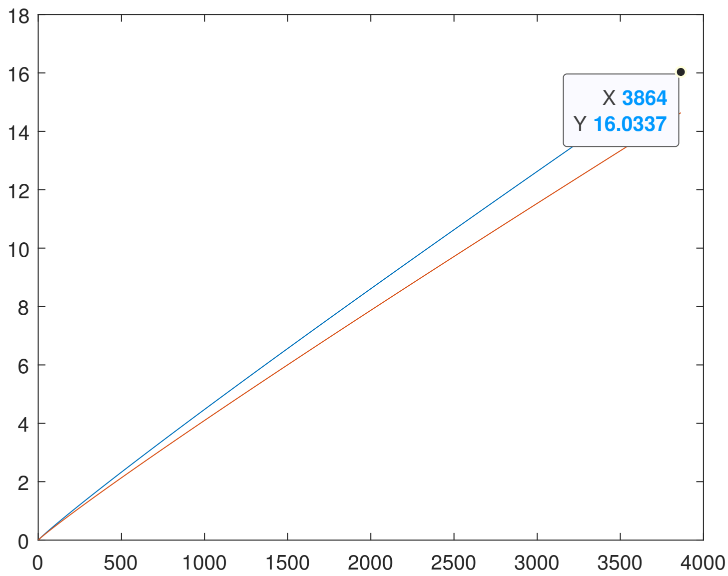
Overview of Esfahan's Mobarakeh Steel Company (Zob Ahan) Esfahans Mobarakeh Steel Company, commonly known as Zob Ahan, is one of the largest steel producers in Iran and the Middle East. As a key player in Iran's steel industry, the company is crucial to both the domestic construction sector and the country's steel export market. Its stock is listed on the Tehran Stock Exchange (TSE) under the ticker symbol ZOB and is one of the actively traded stocks due to its strategic importance and the demand for steel in Irans growing infrastructure projects. Zob Ahans stock performance has been influenced by several factors over the years, making it a central focus for investors in Iran:

Global Steel Prices: The company's stock price is closely linked to the fluctuation of steel prices on global markets. Variations in demand, changes in raw material costs, and production capacity all impact the profitability of Zob Ahan and, in turn, its stock performance. Local Demand: Irans focus on infrastructure development and housing construction generates consistent demand for steel products, which supports the stock price of Zob Ahan. Economic Sanctions: Sanctions on Iran have posed challenges for exports and access to global markets, adding to the volatility of Zob Ahans stock. Government Policies: Steel industry regulations, taxation policies, and government support for infrastructure projects also play a significant role in the companys financial performance and its stock price. Options Trading in Iran and Zob Ahan. Though options trading on the Tehran Stock Exchange (TSE) is relatively new, it has seen increasing interest, particularly for large companies like Zob Ahan.

Zob Ahan options, like other options in the Iranian market, come with a range of



(a) The solution $(V(S,t))$.

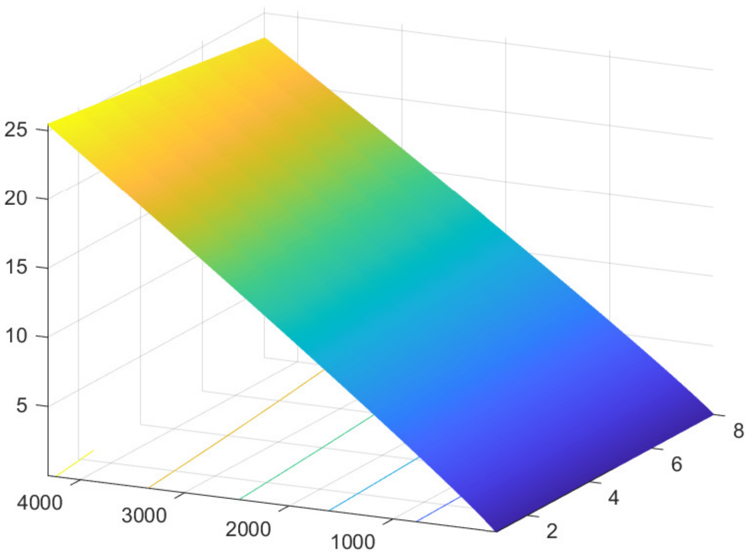


(b) The solution of IKCO $(V(S))$ in $t = 0.1$ and $t = 0.2$.

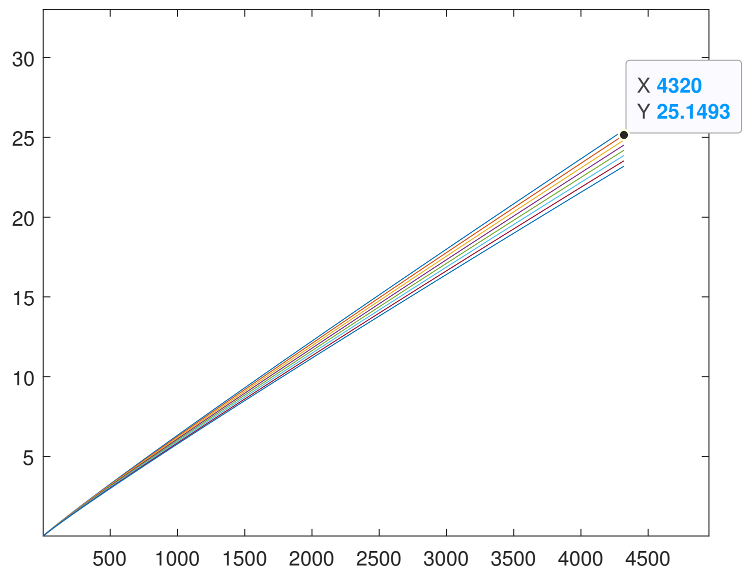
Figure 2: Numerical solution of IKCO.

strike prices and expiration dates. Investors choose based on their market outlook and risk tolerance. Benefits of Options Trading for Zob Ahan Shares includes the following.
Risk Management: By using put options, investors can protect their Zob Ahan share holdings against significant price drops or market downturns.
Income Generation: Selling call options on Zob Ahan shares through a covered call strategy can generate income while allowing the investor to hold the shares for the long term.
Strategic Flexibility: Options provide traders with flexibility to employ different strategies, such as straddles, spreads, or protective puts, to capitalize on various market conditions.
Volatility Management: In periods of heightened volatility, options trading offers a way to profit from price fluctuations without directly owning Zob Ahan shares.
Zob Ahan, as a leading player in Irans steel industry, is of significant interest to investors. Its stock performance is influenced by global steel prices, local demand, and government policies. With the introduction of options trading in Iran, investors now have more tools to manage their positions in Zob Ahan, using options to hedge risks, generate income, or speculate on stock price movements. While the market for options is still evolving, the increasing availability of financial instruments like options provides both opportunities and challenges for traders in Iran's developing financial landscape.

While options trading for Zob Ahan is gaining momentum, one of the key challenges remains liquidity. The options market in Iran is still developing, which can lead to larger spreads between the bid and ask prices and less active trading compared to traditional stocks. To further test the convergence of the algebraic system obtained in this study, the example considered for Zob Ahan stocks has a longer time period. This example studies the securities labeled with code zezob4000 (Zob Ahan Option-200-1404/4/25). The annual volatility of this stock is 2.23568, with a sigma value of 0.34993, the stock price is 432 IRR (Iranian Rial), the strike price is 200 IRR, and the time period is considered as 0.35 of the total period. The interest rate in Iran for the year 1404 (2025) was announced at 30%. To calculate the call value for this stock, the algebraic system derived from the Black-Scholes model, with a time step of 0.05 and a price step of 0.01, was solved. Of course, the final time to solve this problem is set to 0.4 and the values obtained at time 0.35 are the optimal solution for this example. Figure 3a presents the solution of this system for all time and price intervals in this example. Figure 3b shows these values for $t = 0.05, 0.1, 0.15, 0.2, 0.25, 0.30, 0.35, 0.4$, and Table 3, presents the prices obtained at 20 points corresponding to the figure. As can be seen in the table and in Figure 3b, the calculated call price is 251.49 IRR, which shows a negligible difference from the exact value obtained from solving the Black-Scholes equation, which is 251.93 IRR.



(a) The solution $(V(S,t))$.



(b) The solution of NICICO $(V(S))$ in $t = 0.05, 0.1, 0.15, 0.2, 0.25, 0.30, 0.35, 0.4$.

Figure 3: Numerical solution of Zob Ahan.

Table 3: Numerical results for Zob Ahan

time	$V(S_i) \times 10^1$						
$t = 0.05$	0.008	1.421	2.738	4.020	5.286	6.531	7.763
	8.990	10.202	11.407	12.611	13.803	14.990	16.177
	17.354	18.527	19.702	20.868	22.030	23.195	
$t = 0.1$	0.008	1.432	2.764	4.063	5.346	6.608	7.858
	9.103	10.334	11.558	12.780	13.991	15.197	16.403
	17.599	18.792	19.985	21.170	22.351	23.535	
$t = 0.15$	0.008	1.453	2.804	4.121	5.422	6.702	7.970
	9.232	10.481	11.722	12.961	14.190	15.413	16.636
	17.849	19.058	20.269	21.470	22.669	23.869	
$t = 0.20$	0.008	1.473	2.842	4.178	5.497	6.794	8.079
	9.359	10.625	11.883	13.140	14.385	15.625	16.865
	18.095	19.321	20.548	21.766	22.981	24.197	
$t = 0.25$	0.008	1.492	2.880	4.233	5.570	6.885	8.187
	9.484	10.767	12.042	13.315	14.577	15.833	17.090
	18.336	19.578	20.822	22.056	23.287	24.520	
$t = 0.30$	0.008	1.512	2.918	4.288	5.642	6.974	8.293
	9.607	10.906	12.197	13.487	14.765	16.038	17.311
	18.573	19.832	21.091	22.342	23.588	24.837	
$t = 0.35$	0.009	1.531	2.954	4.342	5.713	7.062	8.397
	9.728	11.043	12.351	13.657	14.951	16.239	17.528
	18.807	20.081	21.356	22.622	23.885	**25.149**	
$t = 0.40$	0.009	1.549	2.990	4.395	5.783	7.148	8.500
	9.846	11.178	12.501	13.823	15.133	16.437	17.742
	19.036	20.326	21.617	22.898	24.176	25.456	

4 Stability, Convergence, and Error Analysis of the Proposed Method

4.1 Stability Analysis

In this study, a combined numerical method integrating fully implicit and explicit finite difference schemes is proposed for solving the Black-Scholes PDE. The fully implicit scheme is well-known for its unconditional stability, meaning the numerical solution remains stable regardless of the choices for time and space discretization steps. This property is essential for accurately modeling the evolution of option prices over time without introducing numerical instabilities.

4.2 Convergence Analysis

Convergence of the numerical solution is demonstrated by comparing the computed option prices with the exact analytical solutions of the Black-Scholes equation. As detailed in the manuscript, the numerical results for call option pricing of NICICO, Iran Khodro, and Zob Ahan stocks show minimal deviation from exact values. This close agreement across different test cases validates the convergence of the method as the discretization parameters (time and price steps) are refined.

4.3 Rigorous Error Analysis

- **Truncation Error:** The combined numerical approach effectively reduces truncation errors, which are inherent in finite difference approximations. By integrating both implicit and explicit schemes, the method leverages the stability of the implicit approach and the simplicity of the explicit method, minimizing the overall numerical error.

- **Discretization Steps:** The use of carefully selected time and price steps (as described in the numerical experiments) further minimizes discretization errors. Adaptive refinement of these steps, when necessary, enhances the accuracy of the computed option prices.

- **Validation Against Exact Solutions:** The manuscript provides quantitative comparisons between the numerical and exact solutions. For example, the calculated call prices for the studied stocks differ negligibly from the exact values, confirming both the accuracy and reliability of the proposed scheme.

- **Supporting Table:** To further demonstrate the accuracy of the proposed method, Table 4 presents a comparison between the numerical and exact results for different stocks, highlighting the low observed absolute errors.

Table 4: Numerical and Exact Results

Stock	Numerical Price	Exact Price	Absolute Error
NICICO	4130.7	4131.1	0.4
Iran Khodro	1603.4	1603.7	0.3
Zob Ahan	251.49	251.93	0.44

In conclusion, the proposed combined numerical method demonstrates robust stability and convergence properties, making it a reliable tool for option pricing in emerging markets with complex economic conditions. The rigorous error analysis, supported by validation against exact solutions, underscores the methods accuracy and practical applicability for real-world financial modeling.

5 Conclusion

This study demonstrates the successful application of a combined numerical method for solving the Black-Scholes equation in the context of Tehran's stock market, specifically for call option trading on Iranian Copper Company, Iran Khodro and Zob Ahan stocks. By implementing a numerical scheme with carefully selected time and price steps, the method provided accurate approximations for the call option values. The comparison between the calculated values and the exact solution of the Black-Scholes equation shows minimal deviation, validating the effectiveness of the proposed approach. This combined numerical method proves to be a reliable tool for pricing options in the Tehran Stock Exchange, offering traders and financial analysts a practical solution for handling real-world option pricing problems, especially in markets with unique economic conditions, such as the fluctuating interest rates and volatility observed in Iran. The findings of this research could contribute to more informed decision-making in option trading and enhance the understanding of derivative pricing in emerging markets. Beyond the theoretical validation, it is also essential to acknowledge the practical challenges of applying such models in real-world trading environments, particularly in Irans market. Options markets are known as effective tools for managing risk and increasing returns in world stock exchanges. These tools have also been considered in the Tehran Stock Exchange, but restrictions such as the "open position ceiling" in options contracts have brought challenges. Filling this ceiling has limited trading and reduced market liquidity, which can ultimately lead to a decrease in investor confidence. But what are the factors that cause the problem of filling the open position ceiling? The open position ceiling in options contracts is the maximum number of contracts that can remain open at any given time. When the number of these positions reaches the specified ceiling, it is not possible to open new positions and only current positions can be settled or closed. This reduces trading activities, increases volatility, and new investors exit the market. Ultimately, market liquidity decreases and market performance becomes unstable. International Solutions for Managing Open Position Caps International markets that have more experience in the area of trading options have faced similar challenges and have used various solutions to manage this problem, which will be mentioned.

- **Temporary increase in open position caps:** When market demand is high or volatility has reached its peak, some international exchanges such as Chicago and Europe have temporarily increased open position caps to prevent trading from stopping and reducing liquidity. Gradual entry of new traders: In some advanced markets, after reaching the open position cap, new traders are allowed to enter by reviewing and adjusting the cap. This solution helps the market develop and maintains liquidity.

- **Advanced risk management systems:** International exchanges use advanced risk management systems that provide regulatory authorities with the necessary warnings about the filling of open position caps and prevent potential crises. Implementing such systems in Iran can significantly help improve market efficiency.

With this in mind, some solutions can help the Tehran Stock Exchange in this regard. Gradually increasing the open position limit, strengthening the educational infrastructure for traders, and implementing advanced risk management systems are among the solutions that can make changes in this area. To deal with the problems caused by the open position limit, it is suggested that this limit be increased gradually and according to market conditions. This measure can prevent market lock-up and improve liquidity. In addition, traders' awareness of how derivatives and trading options work is of great importance. It also seems that providing training courses and extensive information can help better manage risk and reduce risks in this market. The use of advanced risk management systems will help the Iranian market to automatically identify fluctuations and risks related to filling the ceiling of open positions and take the necessary measures in a timely manner. These systems should provide the necessary warnings for quick and accurate decision-making by analyzing real-time market data.

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