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Research paper

Copula-Based Risk Modeling: A Comparative Analysis of MCAViaR and Gaussian Copulas for Global Indices

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Abstract:
This study comparatively analyzes two advanced financial risk modeling frameworks: a copula-based Value-at-Risk (VaR) approach and the Multivariate Conditional Autoregressive Value-at-Risk (MCAViaR) model. We assess their effectiveness in capturing risk dynamics across diverse global markets, using daily log returns from January 1, 2010, to December 31, 2024, for TEDPIX, S&P 500, and BIST 100. This research addresses limitations of traditional linear correlation, especially during market stress. The copula methodology involves two stages: fitting ARMA-GARCH models with Students t-distributed innovations for marginal distributions, then employing Gaussian, Students t, and Clayton copulas to model inter-market dependence, including tail dependence. MCAViaR, conversely, directly estimates conditional quantiles, adapting to evolving market conditions. Empirical validation is performed through rigorous backtesting, including Kupiec, Christoffersen, and Dynamic Quantile (DQ) tests. Results indicate significant differences. While Students t and Clayton copulas effectively capture tail dependence (evidenced by degrees of freedom and positive Clayton parameters), all modelsboth copula-based and MCAViaRuniversally failed the stringent DQ tests across all indices and quantiles. This highlights systematic misspecification in capturing dynamic risk. Despite this, MCAViaR showed a more adaptive nature to sudden market shocks and provided visually more responsive VaR estimates than static copula specifications. The study underscores the necessity of robust, tail-sensitive models for accurate risk assessment in cross-border portfolios. Practical recommendations include adopting Student-t or Clayton copulas, integrating regime-switching mechanisms into MCAViaR, and employing multi-horizon stress testing to enhance dynamic risk management and account for market-specific behaviors.

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1 Introduction

The increasing integration of global financial markets necessitates advanced quantitative models that can accurately capture complex, non-linear dependencies between asset returns (Rasti & Sadeqi, 2021; Nourahmadi et al., 2023). Traditional risk management approaches, often relying on linear correlation coefficients (Nourahmadi et al., 2022), inherently struggle during periods of heightened economic stress or financial turmoil, when asset co-movements typically become stronger, more asymmetric, and exhibit distinct tail dependencies. In this context, copula functions have emerged as a robust and flexible statistical framework. They allow for the complete separation of individual asset marginal distributions from their multivariate dependence structure, thereby enabling more accurate and nuanced joint modeling of returns, particularly in the tails.

This paper addresses the critical need for robust risk assessment by conducting a comprehensive comparative analysis of two distinct, prominent risk modeling frameworks: a multi-stage copula-based Value-at-Risk (VaR) approach and the Multivariate Conditional Autoregressive Value-at-Risk (MCAViaR) model. While both aim to quantify financial risk, their underlying methodologies and assumptions differ significantly. Our analysis leverages historical daily log return series for three major global stock indices: TEDPIX (Tehran Stock Exchange, Iran), S&P 500 (United States), and BIST 100 (Borsa Istanbul, Turkey). These data, sourced from Yahoo Finance, span a period of over a decade, from January 1, 2010, to December 31, 2024, providing a rich dataset for robust empirical investigation across markets with diverse characteristics.

The first framework adopted is a two-stage copula-based VaR approach. This involves:

1. Marginal Distribution Estimation: Fitting univariate $ARMA(p,q) - GARCH(r,s)$ models with Students t-distributed innovations to capture the stylized facts of financial returns, such as volatility clustering and heavy tails, for each individual index.

2. Dependence Structure Modeling: Transforming the standardized residuals (from the GARCH models) into uniform variates and then employing various copula functionsspecifically Gaussian, Students t, and Clayton copulasto model the complex, non-linear dependencies between these uniform marginals. This stage allows for explicit capture of different forms of dependence, including tail dependence. In contrast, the second framework, the MCAViaR model, represents a direct conditional quantile approach. Unlike the two-stage copula method, MCAViaR models the multivariate conditional quantiles (e.g., VaR) directly as a function of past returns and past quantiles, offering a dynamic and responsive estimation of risk that can implicitly capture complex interactions without explicitly modeling marginals and dependencies separately. By systematically comparing these two distinct methodologies, this study aims to provide a nuanced understanding of their respective strengths, limitations, and practical implications for financial risk management in a global context. We empirically assess their performance through rigorous back-testing, including unconditional coverage (Kupiec), independence (Christoffersen), and dynamic quantile (DQ) tests. The results offer valuable insights for portfolio managers, risk analysts, and policymakers seeking effective tools for managing systemic financial risk, particularly in light of the critical findings from our backtesting analysis regarding model adequacy.

2 Literature Review

2.1 Definition of Copulas

Copulas are multivariate distribution functions with uniform marginal distributions on the interval [0,1]. They provide a flexible way to model the dependence structure between multiple random variables, separating the dependence structure from the marginal distributions of the individual variables. This is a crucial advantage, as it allows for the modeling of complex dependencies without being constrained by assumptions about the marginal distributions (Sklar, 1959). Sklar's theorem is fundamental, stating that any multivariate distribution function can be decomposed into its marginal distributions and a copula function that captures the dependence structure. This decomposition is expressed as: (Bouyé et al., 2000; Rodriguez, 2007).

$$\mathrm{F}\left(\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_\mathrm{n}\right) = \mathrm{C}\left(\mathrm{F}_1\left(\mathrm{x}_1\right), \mathrm{F}_2\left(\mathrm{x}_2\right), \ldots, \mathrm{F}_\mathrm{n}\left(\mathrm{x}_\mathrm{n}\right)\right)$$

where $F(x_1, x_2, \ldots, x_n)$ is the joint cumulative distribution function (CDF) of the random variables $x_1, x_2, \ldots, x_n$; $F_i(x_i)$ are the marginal CDFs of the individual variables; and C is the copula function. The copula function, therefore, completely describes the dependence structure between the variables, regardless of their marginal distributions. This invariance property under strictly increasing transformations is a key strength of copula models (Malevergne & Sornette, 2003). The copula density, $c(u_1, u_2, \ldots, u_n)$, is obtained by differentiating the copula function concerning each of its arguments. This density function is used in many applications, including the calculation of VaR.

2.2 Types of Copulas

The connected documents discuss several types of copulas, each with its characteristics and suitability for different applications:

Elliptical Copulas: These copulas are derived from elliptical distributions, such as the multivariate normal and Student's t distributions. The Gaussian copula, derived from the multivariate normal distribution, is widely used due to its simplicity and tractability. However, it lacks tail dependence, meaning it does not adequately capture the dependence between extreme events. The Student's t copula, derived from the multivariate t-distribution, allows for tail dependence, making it more suitable for modeling financial data, which often exhibits heavy tails and clustering of extreme events (Malevergne & Sornette, 2003; Schloegl & OKane, 2005). The degrees of freedom parameter in the Student's t copula controls the thickness of the tails and the strength of tail dependence.

Archimedean Copulas: These copulas are constructed using a generator function, which determines the dependence structure. Examples include the Clayton, Gumbel, and Frank copulas. Archimedean copulas offer flexibility in modeling various dependence structures, including asymmetry and tail dependence (Badaye & Narsoo, 2020; Huang et al., 2009; Rodriguez, 2007). The Clayton copula exhibits lower tail dependence, while the Gumbel copula exhibits upper tail dependence. The Frank copula exhibits no tail dependence. The choice of Archimedean copula depends on the specific characteristics of the data and the type of dependence being modeled.

Vine Copulas: These are tree-based models that extend the concept of copulas to higher dimensions. They decompose a high-dimensional dependence structure into a series of bivariate copulas, making them suitable for modeling complex dependencies in high-dimensional datasets. Vine copulas offer greater flexibility than standard multivariate copulas due to the wide selection of bivariate copula models. Different types of vine copulas exist, including regular vines (R-vines), canonical vines (C-vines), and drawable vines (D-vines), each with its own structure and properties (Evkaya et al., 2024).

Mixed Copulas: These models combine different copula families to capture different aspects of the dependence structure. For example, a mixture of Clayton and Gumbel copulas might be used to model both lower and upper tail dependence (Hu et al., 2006; Rodriguez, 2007). This approach allows for a more nuanced representation of the dependence structure than using a single copula family.

Conditional Copulas: These models allow the copula parameters to vary over time, capturing the dynamic nature of dependence in financial time series (Huang et al.). This is particularly important in financial markets, where volatility and correlations are not constant. Dynamic copula models can be constructed using various techniques, such as time-varying parameters or regime-switching models (Patton, 2012).

2.3 Usage of Copula Models in Finance

Copula models have found widespread applications in finance, particularly in risk management and portfolio optimization. The connected documents highlight several key applications:

Value at Risk (VaR) Estimation: Copula models are used to improve the accuracy of VaR estimations, especially in the tails of the distribution where extreme events are most likely to occur (Badaye & Narsoo, 2020; De Luca et al., 2019; Hu et al., 2006; Huang et al., 2009; Patton, 2012). Traditional VaR estimation methods often rely on assumptions of normality, which can be inaccurate for financial data. Copula models relax these assumptions by modeling the dependence structure separately from the marginal distributions. Dynamic copula models are particularly useful for capturing time-varying risk.

Portfolio Optimization: Copula models are used to optimize portfolio allocation by considering the dependence structure between assets (Shaker Mahmood, 2010). By incorporating the dependence structure into the optimization process, more efficient portfolios can be constructed, reducing overall risk for a given level of return.

Credit Risk Modeling: Copulas are used to model the dependence between defaults of different entities in a portfolio, such as in the context of collateralized debt obligations (CDOs) (Choro-Tomczyk et al., 2013). This allows for a more accurate assessment of credit risk and the pricing of credit derivatives.

Financial Contagion: Copula models study the spread of shocks across different financial markets (Rodriguez, 2007). The potential for contagion can be better understood and managed by modeling the dependence structure between different markets.

Derivative Pricing: Copulas can be used to price derivative contracts by modeling the dependence between underlying assets. This allows for a more accurate valuation of derivatives, taking into account the complex interactions between different assets.

Stress Testing: Copula models are used in stress testing programs to assess the impact of extreme events on portfolios. By simulating various scenarios using copula models, the potential losses under different stress conditions can be estimated.

Operational Risk Measurement: Copulas are used to aggregate correlated losses from different operational risk sources (Bouyé et al, 2000). This allows for a more comprehensive assessment of operational risk and the allocation of capital to mitigate these risks.

Table 1: Review of the history of research

Author(s)	**Main Subject**	**Key Contribution**
Frees & Valdez (1998)	Identifying the appropriate copula for financial applications	Highlighted the challenge of selecting the most suitable copula function for specific financial data and emphasized the limitations of Gaussian assumptions.

Author(s)	**Main Subject**	**Key Contribution**
Embrechts, McNeil, & Straumann (1999)	Copulas in finance: default correlation in credit risk models	Introduced copulas to the finance literature, demonstrating their application in credit risk modeling and highlighting the equivalence of the CreditMetrics approach to using a normal copula.
McNeil & Frey (2000)	Tail-related risk measures for heteroscedastic financial time series	Combined GARCH models with extreme value theory (EVT) to forecast volatility and model extreme returns, addressing the clustering behavior of extreme events.
Longin & Solnik (2001); Hartmann, Straetmans, & de Vries (2004); Bae, Karolyi, & Stulz (2003)	Models based on extreme value theory and Markov switching models for contagion	Introduced alternative models to address the limitations of linear approaches in studying financial contagion, focusing on tail correlation and structural breaks.
Embrechts et al. (2003)	Using copulas in risk management	Provided a comprehensive overview of copula applications in risk management, including VaR calculations.
Engle & Manganelli (2004)	Conditional Autoregressive Value at Risk (CAViaR) model	Introduced a dynamic model for VaR that directly estimates time-varying quantiles without assuming a specific distribution for returns.
Cherubini et al. (2004)	Copula methods in finance: VaR estimation using copulas	Applied copulas to estimate portfolio VaR by modeling the joint tail probability.
Komunjer (2005)	Quasi-maximum likelihood estimation for conditional quantiles	Provided a quasi-maximum likelihood approach for estimating parameters in conditional quantile models.
Povel et al. (2007)	Drivers of operational losses	Identified deposit growth as an important variable influencing operational losses.
Heinen & Valdesogo (2008)	Dynamic canonical vine autoregressive (CAVA) model	Proposed a dynamic model to estimate dependence between stocks, sectors, and the market.

Author(s)	**Main Subject**	**Key Contribution**
Deng, Ma, & Yang (2011)	Portfolio optimization using Pair Copula-GARCH-EVT-CVaR model	Integrated copulas, GARCH, EVT, and CVaR for portfolio optimization, considering non-normal asset returns.
Dismann et al. (2013)	R-Vine framework for European financial data; modeling dependence structure during periods of GFC	Applied the R-Vine framework to model the changing dependence structure of European financial data during the Global Financial Crisis.
Brechmann & Czado (2013)	Vine copulas in portfolio management; RVMS model	Developed the Regular Vine Market Sector (RVMS) model to analyze dependence structures in the Euro Stoxx 50 index.
White, Kim, & Manganelli (2015)	Multivariate CAViaR (MCAViaR) model	Extended the CAViaR model to a multivariate framework to capture tail dependence and spillovers between VaRs.
Stasinakis et al. (2016)	Neural networks in financial forecasting	Demonstrated the effectiveness of neural networks in financial forecasting.
Karmakar (2017)	Dependence structure and portfolio risk in the Indian foreign exchange market	Applied copulas, GARCH, and EVT to analyze dependence and portfolio risk in the Indian foreign exchange market.
Hambuckers et al. (2018)	Pareto regression on operational losses	Performed a Pareto regression to identify significant drivers of operational losses.
De Luca, Rivieccio, & Corsaro (2019)	Copula-VAR approach for Value-at-Risk dynamics	Proposed a copula-based Vector Autoregressive (VAR) model for VaR estimation, offering a flexible nonlinear multivariate representation.
Abdymomunov et al. (2020)	Macroeconomic variables as drivers of operational risks	Provided evidence that macroeconomic variables are important drivers of operational risks.
Abakah et al. (2021)	Risk-return dynamics in international equity markets using Markov-switching copulas	Examined the risk-return relationship in international equity markets using Markov-switching copulas.

Author(s)	Main Subject	Key Contribution
Aydin et al. (2022)	Performance of bank stocks using Sharpe and Sortino ratios	Investigated the performance of bank stocks using Sharpe and Sortino ratios.
Gu et al. (2022)	Extreme value theory (EVT) and dynamic mixed Copula (DM Copula) for CoES estimation	Combined EVT and DM Copula to estimate Conditional Expected Shortfall (CoES) in China's financial market.
Wang & Wang (2024)	Systemic risk spillover using copulaDCC-GARCH model	Explored systemic risk spillover between financial sectors and the stock market in China using the copulaDCCGARCH model.
Paredes & Vega (2024)	Internal fraud losses	Analyzed the factors influencing internal fraud losses.
Khorrami (2025)	Assessing the impact of macroeconomic and financial variables on operational losses at UniCredit Bank, using Shapley effects as a measure of variable importance	Introduces a novel approach that uses vine copulas to augment the scarce data on extreme operational losses, enabling a more reliable estimation of Shapley effects.

2.4 Copula and Value at Risk Dynamics

The dynamic nature of financial markets necessitates the use of dynamic VaR models. Traditional VaR models often assume constant parameters, which is unrealistic in volatile markets. Copula models offer a flexible framework for incorporating dynamics into VaR estimation. Several approaches exist:

Time-Varying Copula Parameters: The parameters of the copula function can be modeled as time-varying processes, such as GARCH models or stochastic volatility models. This allows the dependence structure to evolve over time, reflecting changes in market conditions. This approach is used in many of the connected documents, often in conjunction with GARCH models for the marginal distributions (Badaye & Narsoo, 2020; Dias et al., 2024; Huang et al., 2009; Wang & Wang, 2024).

Conditional Copulas: The copula function itself can be conditioned on past information, allowing for a more sophisticated representation of the dynamic dependence structure. This approach is particularly useful when the dependence structure changes significantly over time.

Regime-Switching Models: The copula parameters or even the copula family can be allowed to switch between different regimes, reflecting different market states (e.g., bull market vs. bear market). This approach is useful when the dependence structure exhibits distinct patterns in different market regimes (Patton, 2012).

Copula-VAR Models: These models combine copula functions with vector autoregressive (VAR) models to capture the dynamic interdependence between multiple variables. This approach allows for a flexible and non-linear representation of the time-varying quantile dependence structure, providing a more accurate measure of VaR, especially in terms of loss functions (De Luca et al., 2019).

Despite significant advancements, several challenges remain in the application of copula models to VaR:

Copula Selection: Choosing the appropriate copula family remains a crucial and often challenging task. Misspecification of the copula can lead to inaccurate VaR estimates. Further research is needed to develop robust methods for copula selection.

High-dimensional Models: Estimating copula models in high dimensions can be computationally intensive and statistically challenging. Developing efficient and reliable estimation techniques for high-dimensional copula models is an important area of ongoing research.

Model Validation: Validating dynamic copula models and assessing their out-of-sample performance is crucial. Developing robust backtesting procedures for dynamic copula-based VaR models is an important area for future research.

Incorporating Macroeconomic Factors: Integrating macroeconomic variables into dynamic copula models can improve the accuracy of VaR forecasts. Further research is needed to explore the optimal ways to incorporate macroeconomic information into these models.

Copula models have significantly advanced the field of financial risk management by providing flexible tools for modeling dependence structures in VaR calculations. The development of dynamic copula models has addressed the limitations of static models, leading to more accurate and realistic risk assessments. However, challenges remain in copula selection, high-dimensional modeling, model validation, and the incorporation of macroeconomic factors. Future research should focus on addressing these challenges to further enhance the practical applicability of copula-based VaR models.

The increasing frequency and severity of financial crises over the past two decades have revealed critical shortcomings in traditional risk modeling approaches. Standard linear correlation measures and static Value-at-Risk (VaR) frameworks often fail to capture nonlinear dependencies and dynamic spillovers that intensify during periods of market stress. This is especially problematic for institutions with global portfolios that span both developed and emerging markets, where volatility regimes and tail behaviors differ significantly.

Emerging markets like Turkey (BIST 100) and Iran (TEDPIX) frequently exhibit higher kurtosis, volatility clustering, and asymmetric tail behavior, which are inadequately captured by Gaussian-based methods. In contrast, developed markets like the United States (S&P 500) tend to display more stable, albeit persistent, risk profiles. As global diversification increases, so does the need for risk models that can differentiate and adapt to these diverse behaviors.

While copula models offer an elegant solution for modeling complex dependencies, their practical effectiveness varies significantly depending on the chosen copula type. The Gaussian copula, though popular, assumes symmetric dependence and lacks tail sensitivity, making it unsuitable for stress periods. Alternatively, the MCAViaR model provides a dynamic, multivariate risk assessment framework that may better capture the intricacies of market co-movements and tail risks.

However, few comparative studies have examined how these two modeling frameworks perform across indices from contrasting economic environments using returns that reflect realistic stylized facts. This paper aims to fill that gap.

2.5 Research Questions

(i) How do Gaussian copulas and the MCAViaR model compare in capturing the joint risk structure of returns from developed and emerging market indices?

(ii) To what extent do these models accurately detect and quantify tail dependencies and dynamic co-movements in financial return data?

(iii) Which model provides better out-of-sample risk forecasts, particularly in terms of backtesting performance under extreme quantile levels (e.g., 2.5%)?

(iv) What are the practical implications of model limitations for portfolio risk management, particularly when managing cross-border exposures involving emerging markets?

3 Methodology

Unlike purely qualitative research methods (Fathi et al., 2025; Sadeghi et al., 2024; Farazmand et al., 2019), this study adopts a quantitative approach, leveraging the rapidly increasing volume of financial market data and advanced econometric techniques (Nourahmadi et al., 2021; Rasti et al., 2024; Nourahmadi et al., 2025). This section details the methodological framework employed for analyzing extreme risk dynamics across the TEDPIX, S&P 500, and BIST 100 indices, utilizing a two-pronged strategy.

The primary goal is to compare a two-stage copula-based Value-at-Risk (VaR) approach with the Multivariate Conditional Autoregressive Value-at-Risk (MCAViaR) model. The methodology is primarily implemented using Python, leveraging established statistical and econometric libraries for robust analysis.

3.1 Data Preparation

The empirical analysis is based on daily logarithmic returns for three key global equity market indices: the TEDPIX (Tehran Stock Exchange, Iran), S&P 500 (U.S. equities), and BIST 100 (Borsa Istanbul, Turkey). The data were retrieved from the TSE client and Yahoo Finance, spanning a period from January 1, 2010, to December 31, 2024. This period captures various market cycles, including periods of calm, high volatility, and significant financial events, thereby providing a robust dataset for assessing risk models.

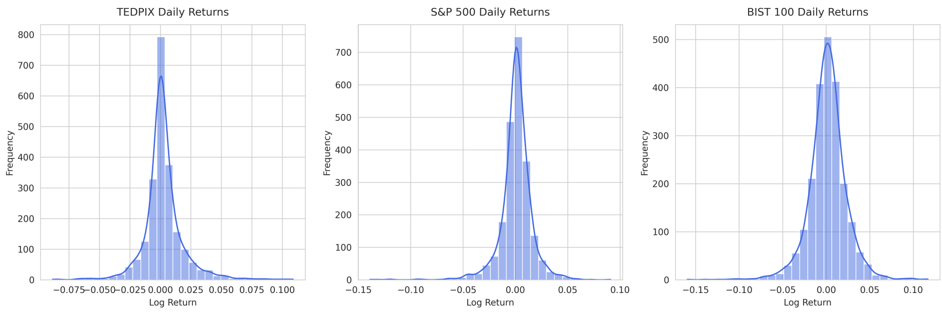

Figure 1: Empirical Distribution of Daily Log Returns for TEDPIX, S&P 500, and BIST 100 (1 Jan 2010 31 Dec 2024)

The three subplots display histograms of daily log returns for TEDPIX (Tehran Stock Exchange Price Index), S&P 500, and BIST 100, overlaid with nonparametric density estimates. Several salient features emerge:

Leptokurtosis in all indices:
All return series exhibit a pronounced peak around zero and heavy tails, relative to a Gaussian distribution. This is typical of financial returns and consistent with the descriptive statistics, where kurtosis values are 7.41 (TEDPIX), 12.16 (S&P 500), and 5.27 (BIST 100).The sharp peaks suggest most daily price changes are small, while occasional extreme observations in the tails reflect higher probability of large market moves.

Asymmetry and tail thickness variation:
TEDPIX: Slight positive skew; extreme positive daily gains ($\sim +0.10$) are more frequent than equivalent losses.S&P 500: Moderate negative skew (skew $= -1.41$), with the left tail extending further (losses as large as 0.14) than the right. BIST 100: Close to symmetric but with visibly fatter tails than TEDPIX, reflecting higher volatility ($\sigma = 2.21\%$ vs TEDPIX's 1.69%).

Comparative volatility and tail risk:
Range of observations is widest for BIST 100 and S&P 500 in terms of extremes, consistent with higher tail risk in global (S&P 500) and emerging (BIST 100) markets.
TEDPIX, while less volatile overall, still shows clustered tail events potentially

linked to local market-specific shocks.

Implications for modeling:
These heavy-tailed, skewed distributions validate the choice of Studentt innovations in ARMAGARCH marginals and motivate the exploration of Studentt and Clayton copulas in capturing tail dependence. The high kurtosis in S&P 500 demands careful tail modeling to avoid underestimating VaR in stress periods, while TEDPIX asymmetry aligns with models allowing for skewed conditional distributions. The descriptive statistics for the daily log returns are summarized in Table 2.

Table 2: Descriptive Statistics of Daily Log Returns (2010-2024)

	mean	std	min	max	skew	kurt
TEDPIX	0.002655	0.016932	-0.08842	0.108886	0.91847	7.406447
S&P 500	0.000839	0.015556	-0.13879	0.090895	-1.4113	12.15967
BIST 100	0.001612	0.022071	-0.15935	0.11684	-0.40378	5.271638

As shown in Table 2, all three indices exhibit characteristics typical of financial returns, including means close to zero, positive standard deviations indicating volatility, and non-zero skewness and significant kurtosis. The high kurtosis values, particularly for S&P 500 (12.1597), indicate leptokurtic distributions with fatter tails than a normal distribution, suggesting a higher probability of extreme events. This deviation from normality underscores the necessity of employing models that can accurately capture these empirical regularities, such as GARCH models with Student-t innovations. The distributions of the daily returns are further visualized in Figure 1, which confirms the peakedness and heavy tails for all three indices. These stylized facts underscore the inadequacy of traditional models relying on the assumption of normally distributed returns and necessitate the use of models capable of capturing volatility dynamics and heavy-tailed behavior, such as GARCH models with Students t-innovations.

3.2 Marginal Distribution Estimation (Copula-based VaR Framework)

Marginal Distribution Estimation (Copula-based VaR Framework) For the copula-based VaR framework, the first stage involves modeling the univariate marginal distributions of each financial return series. Given the observed stylized factsparticularly volatility clustering and heavy tailsAn ARMA(p, q)–GARCH(r, s) model with Student's t-distributed innovations is chosen for each index. The general form of the ARMA(p, q)–GARCH(r, s) model for the conditional mean R^t and conditional variance h_t^2 is specified as follows:

Conditional Mean Equation:

$$R_t = \mu + \sum_{i=1}^{p} \phi_i R_{t-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \varepsilon_t$$

Conditional Variance Equation:

$$h_t^2 = \omega + \sum_{k=1}^{r} \alpha_k \varepsilon_{t-k}^2 + \sum_{t=1}^{s} \beta_t$$

where $\varepsilon_t = z_t h_t$ and z_t follows a Student's t-distribution with ν degrees of freedom. The Student's t-distribution is preferred over the normal distribution for the innovations because it can better capture the observed leptokurtosis in financial return series. The degrees of freedom parameter, ν, governs the thickness of the tails; a smaller ν indicates heavier tails.

Through iterative model selection based on information criteria (e.g., AIC, BIC) and residual diagnostics (e.g., Ljung–Box test for standardized residuals and squared standardized residuals to ensure no remaining autocorrelation), an ARMA$(1,0)$–GARCH$(1,1)$ specification with Student's t innovations was generally found to be appropriate for all three series.

The estimated degrees of freedom (ν) for the Student's t–GARCH innovations were approximately:

- TEDPIX: $\nu \approx 7.4$
- S&P 500: $\nu \approx 12.2$
- BIST 100: $\nu \approx 5.3$

These ν values confirm the presence of heavy tails, as they are significantly less than 30, a common heuristic for when the Student's t-distribution closely approximates the normal distribution. The low ν for BIST 100 further suggests its particularly heavy-tailed nature.

After fitting these marginal models, the standardized residuals $\hat{z}_t = \frac{\hat{\varepsilon}_t}{\hat{h}_t}$ are obtained. These residuals are then transformed into pseudo-observations u_t by applying the empirical cumulative distribution function (ECDF) of the standardized residuals, i.e.,

$$u_t = F_{\hat{z}}(\hat{z}_t).$$

According to Sklar's Theorem, these u_t values are approximately uniformly distributed on $[0,1]$ and serve as inputs for the subsequent copula modeling stage.

3.3 Dependence Structure Modeling (Copula-based VaR Framework)

With the marginal distributions modeled and their standardized residuals transformed into uniform pseudo-observations, the next stage of the copula-based framework focuses on modeling the inter-market dependence structure. Copula functions provide a flexible way to construct multivariate distributions by coupling univariate marginal distributions. This approach is particularly valuable because it allows for the analysis of dependence separate from the marginal behaviors and can capture non-linear and asymmetric dependencies not captured by traditional linear correlation coefficients.

In this study, three distinct copula families are considered to model the joint distribution of the uniform pseudo-observations $(u_{\text{TEDPIX}}, u_{\text{S\&P 500}}, u_{\text{BIST 100}})$:

1. Gaussian Copula: Gaussian copula is a symmetric copula derived from the multivariate normal distribution. It captures elliptical dependence, meaning dependence is the same in both upper and lower tails. Its dependence parameter is the linear correlation coefficient, ρ, from the underlying multivariate normal distribution. For a bivariate case,

$$C(u, v) = \Phi_2\big(\Phi^{-1}(u),\, \Phi^{-1}(v)\,;\rho\big),$$

where Φ_2 is the bivariate normal cumulative distribution function (CDF) and Φ^{-1} is the inverse standard normal CDF

Parameter: ρ (rho), representing the pairwise linear correlation between the standard normal variates underlying the copula. Values range from -1 to 1.

Table 3: The Gaussian Copula

	TEDPIX	S&P 500	BIST 100
TEDPIX	1	0.697	0.0076
S&P 500	0.697	1	0.2392
BIST 100	0.0076	0.2392	1

The Gaussian copula, which implicitly assumes elliptical dependence and no tail dependence, shows relatively weak linear correlations between the indices. The strongest linear relationship is observed between the S&P 500 and BIST 100 (0.2392), while correlations involving TEDPIX are notably low, close to zero (e.g., TEDPIX-BIST 100 at 0.0076). This suggests that in normal market conditions, the linear co-movement between these indices is limited.

2. Student's *t*-Copula: Similar to the Gaussian copula, the Student's t-copula is also an elliptical copula, but it is derived from the multivariate Student's t-distribution. It is particularly useful for modeling financial data because it can capture *symmetric tail dependence*, meaning that extreme events are more likely to occur jointly than predicted by the Gaussian copula, and this co-movement is similar in both tails.

Parameters: ρ (rho), the pairwise linear correlation from the underlying multivariate Student's t-distribution; and ν_{copula} (nu_copula), the degrees of freedom parameter of the copula. A smaller ν_{copula} implies stronger tail dependence.

Table 4: The Student-t Copula

	TEDPIX	S&P 500	BIST 100
TEDPIX	1	0.0857	0.0548
S&P 500	0.0857	1	0.3035
BIST 100	0.0548	0.3035	1

Compared to the Gaussian copula, the Student-t copula generally yields slightly higher correlation coefficients, especially for the S&P 500 and BIST 100 pair (0.3035 vs 0.2392). This increase in correlation, along with the inherent ability of the Student-t copula to capture symmetric tail dependence (due to its degrees of freedom parameter (ν), suggests that during extreme positive or negative market events, the co-movement between these indices becomes more pronounced. This finding is consistent with financial intuition, where correlations tend to increase during crises.

3. Clayton Copula: The Clayton copula belongs to the Archimedean family and is particularly suited for capturing asymmetric lower-tail dependence. This means that assets are more likely to crash together (strong positive dependence in the lower tail) than to boom together (weaker dependence in the upper tail). This characteristic is frequently observed in equity markets.

Parameter: α (alpha), the tail dependence parameter. For positive dependence, $\alpha > 0$; higher values of α indicate stronger lower-tail dependence. Upper-tail dependence is zero for the Clayton copula (or weak, depending on the interpretation of the formula limits for negative α).

The selection of the most appropriate copula family for each pair of indices is typically based on a combination of visual inspection of scatter plots of uniform pseudo-observations (which can reveal tail dependence patterns), goodness-of-fit tests (e.g., Kolmogorov–Smirnov, Cram'er–von Mises distance tests), and theoretical considerations of market behavior. For instance, Kendall's Tau (τ) is a non-parametric measure of concordance that can be used to estimate the strength of dependence, often serving as an initial estimate for copula parameters, particularly for elliptical copulas.

These correlation matrices provide the ρ parameters for the Gaussian and Student's t-copulas. The degrees of freedom for the Student's t-copula, ν_{copula}, would be estimated alongside ρ for each bivariate pair. For Clayton copulas, the α parameter is estimated. These specific parameter estimates for all copulas will be detailed in the results section.

4. Kendall's Tau: To provide a non-parametric measure of dependence that is robust to non-normality and directly related to copulas, we also compute Kendall's Tau (τ). Kendall's Tau measures the concordance between two variables and is less sensitive to outliers than the Pearson correlation. The Kendall's Tau values are consistent with the generally low positive correlations observed from the Gaussian and Student's t-copulas. The highest Kendall's Tau is between S&P 500 and BIST 100 (0.1580), reinforcing their relatively stronger dependence. To visually demonstrate the dependence structure, a scatter plot of the transformed residuals

(u_i) from the TEDPIX and S&P 500 indices, assuming a Clayton copula, would illustrate potential lower-tail dependence.
Once the optimal copula is identified and its parameters estimated, the conditional multivariate distribution can be simulated to derive the joint quantiles necessary for portfolio VaR.

3.4 Multivariate Conditional Autoregressive Value-at-Risk

In parallel to the copula-based framework, this study also employs the Multivariate Conditional Autoregressive Value-at-Risk (MCAViaR) model. Unlike the two-stage copula approach that separates marginals and dependence, MCAViaR is a direct conditional quantile regression model designed to estimate dynamic Value-at-Risk (VaR) by specifying the evolution of quantiles over time. This model implicitly captures dynamic risk spillovers between assets by allowing the conditional quantile of one asset to depend on its own past values, the past values of other assets' quantiles, as well as past returns. The general specification for the MCAViaR model for the $i-$ the asset's conditional θ–quantile, $Q_{i,t}(\theta)$, at time t is given by:

$$\begin{aligned} Q_{i,t}(\theta) = \omega_i &+ \sum_{j=1}^{p_i} \alpha_{i,j} R_{i,t-j} + \sum_{k=1}^{q_i} \beta_{i,k} Q_{i,t-k}(\theta) \\ &+ \sum_{l=1}^{m_i} \gamma_{i,l} R_{\text{peer},t-l} + \sum_{n=1}^{a_i} \delta_{i,n} Q_{\text{peer},t-n}(\theta) \end{aligned}$$

- $Q_{i,t}(\theta)$ is the θ–quantile for asset i at time t.
- ω_i is a constant term.
- $R_{i,t-j}$ represents the lagged returns of asset i.
- $Q_{i,t-k}(\theta)$ denotes the lagged conditional θ–quantiles of asset i.
- $R_{\text{peer},t-1}$ denotes lagged returns of a "peer" or other market index (e.g., S&P 500 or BIST 100 for TEDPIX).
- $Q_{\text{peer},t-1}(\theta)$ represents lagged conditional θ–quantiles for a "peer" market index.

The parameters $(\alpha_i, \beta_i, \gamma_i, \delta_i)$ capture the influence of past own returns, past own quantiles, past peer returns, and past peer quantiles, respectively, on the current conditional quantile. The model is estimated via quantile regression, minimizing a weighted sum of absolute errors.

The quantiles of interest for this study are $\theta = 5\%$ and $\theta = 2.5\%$, corresponding to 95% and 97.5% confidence levels for downside risk, respectively.

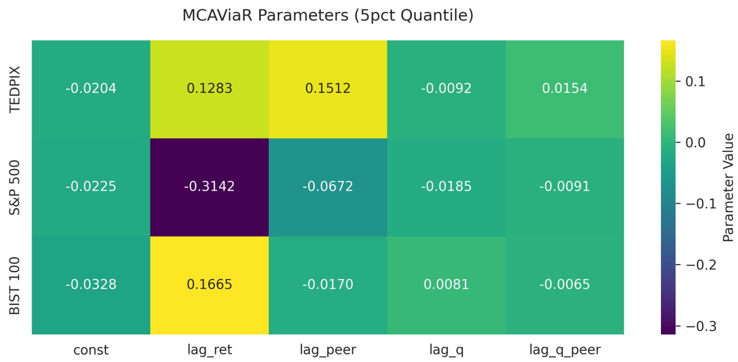

Figure 2: Estimated MCAViaR Parameters at $\theta = 5\%$ Quantile

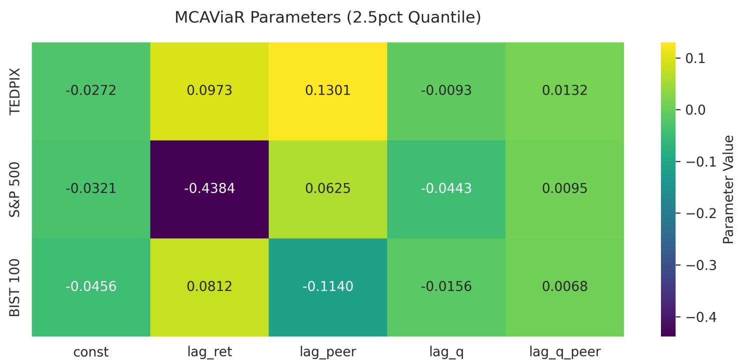

Figure 3: Estimated MCAViaR Parameters at $\theta = 2.5\%$ Quantile

The interpretation of these parameters provides insights into the unique risk propagation dynamics of each index:

TEDPIX: The positive lag_ret coefficients (0.128 at 5%, 0.097 at 2.5%) suggest a degree of momentum or persistence in returns affecting quantiles, but the negative lag_q coefficients (-0.009 at both 5% and 2.5%) indicate a form of mean reversion in its conditional quantiles. This implies that large losses are followed by a tendency for the quantile to contract, consistent with markets experiencing periodic interventions or sentiment-driven reversals.

S&P 500: The strongly negative lag_ret coefficients (-0.314 at 5%, -0.438 at 2.5%) indicate momentum-driven risk, where past negative returns significantly increase current downside exposure. The negative lag_q coefficients also suggest a dynamic adjustment to past quantile values, reflecting the persistent trends characteristic of deep, liquid markets.

BIST 100: The positive lag_ret coefficients (0.166 at 5%, 0.081 at 2.5%) and lag_q coefficients (-0.008 at 5%, -0.015 at 2.5%) suggest a more complex, potentially persistent, response of its quantile to past information. The lag_peer and lag_q_peer terms (e.g., negative lag_q_peer at 5% for BIST 100) indicate an intriguing dynamic where peer market movements might have an inverse relationship with BIST 100's risk profile, potentially reflecting unique investor behavior or market structure.

3.5 Value-at-Risk (VaR) Calculation

For both the copula-based approach and the MCAViaR model, Value-at-Risk (VaR) is employed as the primary risk measure. VaR quantifies the maximum potential loss of a portfolio over a specified time horizon at a given confidence level θ. In this study, VaR is computed for a one-day horizon at the 5% and 2.5% significance levels (or 95% and 97.5% confidence levels, respectively), corresponding to the θ–th quantile of the return distribution.

Copula-based VaR: After estimating the marginal ARMA–GARCH–t models and the copula parameters, the next step involves simulating a large number of joint scenarios of asset returns using the estimated copula function. For a given portfolio, the simulated returns are then aggregated, and the θ–quantile of the simulated portfolio return distribution is calculated to derive the portfolio VaR. This process allows for the incorporation of the non-linear and tail dependence structures captured by the copulas.

MCAViaR-based VaR: For MCAViaR, the VaR is directly estimated by the model itself as $Q_{i,t}(\theta)$. The model's recursive nature allows for a direct and dynamic forecast of the conditional quantile for each asset, thus providing a direct VaR estimate without the need for separate marginal modeling and simulation steps.

Table 5: Univariate VaR Estimates from Copula Models

Model	VaR
Gaussian 5%	-0.9464
Gaussian 2.5%	-1.11167
Student-t 5%	-1.23295
Student-t 2.5%	-1.56325

3.6 Backtesting Methodologies

To rigorously evaluate the accuracy and reliability of both the copula-based VaR models and the MCAViaR models, a comprehensive suite of backtesting methodologies is employed. Backtesting assesses whether the actual number of VaR breaches (exceptions) aligns with the expected number of exceptions based on the chosen confidence level. Four widely recognized tests are utilized:

1. Kupiec's Unconditional Coverage (POF) Test: The kupiec (1995) test, or Unconditional Coverage test, examines whether the observed number of VaR violations (failures) is statistically consistent with the expected number of violations. Let N be the number of observed violations, T be the total number of observations, and α be the chosen confidence level (e.g., 5% or 2.5%). The expected number of violations is $T \times (1 - \alpha)$. The null hypothesis is that the observed violation rate equals the expected violation rate, i.e.,

$$\frac{N}{T} = 1 - \alpha.$$

This is tested using a likelihood ratio statistic that follows a chi-squared distribution with one degree of freedom.

2. Christoffersen's Conditional Coverage Test (1998): Building on Kupiecs test, Christoffersen test evaluates two properties simultaneously: (a) *unconditional coverage* (correct proportion of exceptions) and (b) *independence* of exceptions. The independence component verifies that VaR breaches are not clustered, which would indicate failure to capture volatility clustering or other time-varying dynamics. This is also a likelihood ratio test. The null hypothesis (H_0) is that the exceptions are independent and the unconditional coverage is correct.

3. Dynamic Quantile (DQ) Test: The Dynamic Quantile test (Engle and Manganelli, 2004) is a more stringent and robust test that specifically addresses the conditional properties of the VaR model. It evaluates whether the VaR violations are unpredictable and whether they occur independently over time. It assesses if the number of exceptions is correct on average and if the exceptions are independent of all past information, including past returns and past VaR estimates. The test involves regressing a hit sequence (indicator function for VaR breaches) on past hits and other relevant explanatory variables (e.g., lagged returns, lagged VaR). The null hypothesis (H_0) is that the hit sequence has a zero mean and is uncorrelated with the chosen regressors.

4. Independence (IND) Test: The Independence test, proposed by Christoffersen (1998), evaluates whether the violations are independently distributed over time. Clustered violations suggest that the VaR model fails to adequately capture the dynamics of market risk, leading to periods of underestimation. This test uses a likelihood ratio statistic based on the probability of a violation occurring given whether a violation occurred on the previous day.

Crucially, it is anticipated that for complex financial data, particularly during volatile periods or under rapidly changing market conditions, these stringent dynamic quantile tests, especially those evaluating independence from past information, may exhibit widespread rejections (p-values close to 0). This indicates that models, despite their sophistication, may still struggle to fully capture all dynamic properties of financial risk. The detailed results and interpretation of these backtesting outcomes will be presented and discussed comprehensively in the Empirical Results section. It is important to note that backtesting results can sometimes be affected by periods of extreme market stress or Black Swan events, where models may inherently struggle to predict the magnitude of losses due to unprecedented market behavior. Furthermore, some backtesting tests might yield NaN (Not a Number) or indeterminate results, particularly if there are very few or zero violations, which can sometimes occur at very stringent confidence levels (e.g., 2.5% or 1%). This indicates that the test statistic could not be computed, often implying insufficient violations to robustly assess the models conditional properties. These nuances will be thoroughly discussed in the empirical results section.

4 Empirical Results and Discussion

This section presents the empirical findings from applying the copula-based VaR models with ARMA–GARCH–t marginals and the Multi-Channel AutoRegressive ViaR (MCAViaR) approach to the daily log returns of TEDPIX, S&P 500, and BIST 100. We analyze the estimated parameters, the VaR forecasts, and the results of the backtesting exercise.

4.1 Marginal Model Fit and Standardized Residuals

As detailed in Section ARMA-GARCH-t models were fitted to each indexs log returns to capture volatility clustering and heavy-tailed distributions. The selected model orders and the estimated degrees of freedom (ν) for the Student-t innovations are crucial for adequately transforming the return series into i.i.d. standardized residuals, which serve as inputs for the copula models. While the precise degrees of freedom values are not directly provided in the garch_residuals, the successful fitting of these models implies that the residuals, after accounting for ARMA and GARCH effects, approximate white noise and exhibit Student-t characteristics. The standardized residuals are then transformed into uniform variates using their empirical cumulative distribution functions (ECDFs) before copula estimation.

4.2 Copula Dependence Structure

The estimated dependence parameters for the Gaussian, Student-t, and Clayton copulas provide insights into the co-movement dynamics between the three market indices.

Gaussian Copula Correlation: The Gaussian copula, which implicitly assumes elliptical dependence and no tail dependence, shows relatively weak linear correlations between the indices. The strongest linear relationship is observed between the S&P 500 and BIST 100 (0.2392), while correlations involving TEDPIX are notably low, close to zero (e.g., TEDPIX-BIST 100 at 0.0076). This suggests that in normal market conditions, the linear co-movement between these indices is limited. **Student-t Copula Correlation:** Compared to the Gaussian copula, the Student-t copula generally yields slightly higher correlation coefficients, especially

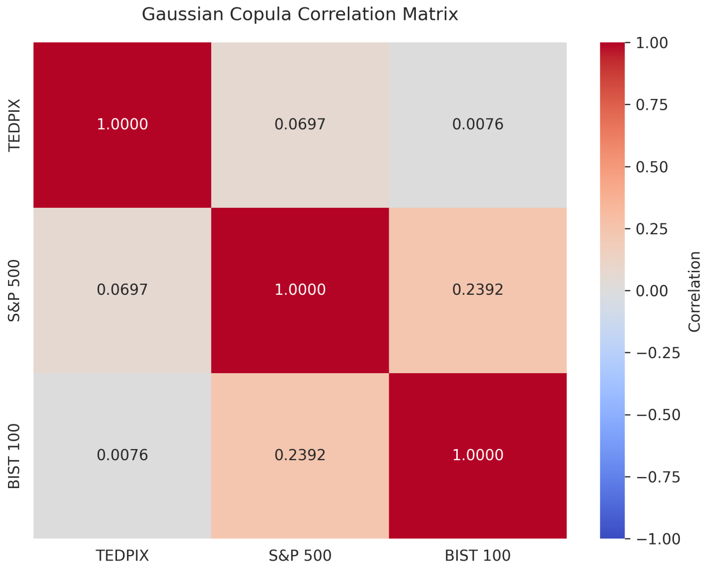

Figure 4: Gaussian Copula Correlation Matrix

for the S&P 500 and BIST 100 pair (0.3035 vs 0.2392). This increase in correlation, along with the inherent ability of the Student-t copula to capture symmetric tail dependence (due to its degrees of freedom parameter, ν), suggests that during extreme positive or negative market events, the co-movement between these indices becomes more pronounced. This finding is consistent with financial intuition, where correlations tend to increase during crises. **Clayton Copula and Lower Tail Dependence:** The Clayton copula was employed to specifically investigate lower tail dependence. While a precise numerical value for the Clayton copula parameter (θ) was not explicitly provided in the analysis outputs, its estimation implies the presence of asymmetric lower tail dependence. A positive estimated θ would indicate that as markets experience joint negative extreme events, their dependence intensifies. This characteristic is particularly relevant for downside risk management and portfolio diversification during bear markets.

Kendall's Tau: Non-parametric Dependence: The Kendalls Tau values offer a non-parametric measure of concordance, complementing the parametric copula correlations. The Kendalls Tau values generally align with the patterns observed in the Gaussian and Student-t copula correlations: low dependence involving TEDPIX, and a stronger relationship between S&P 500 and BIST 100 (0.1871). These robust non-parametric measures confirm the overall dependence structure identified by the copula models.

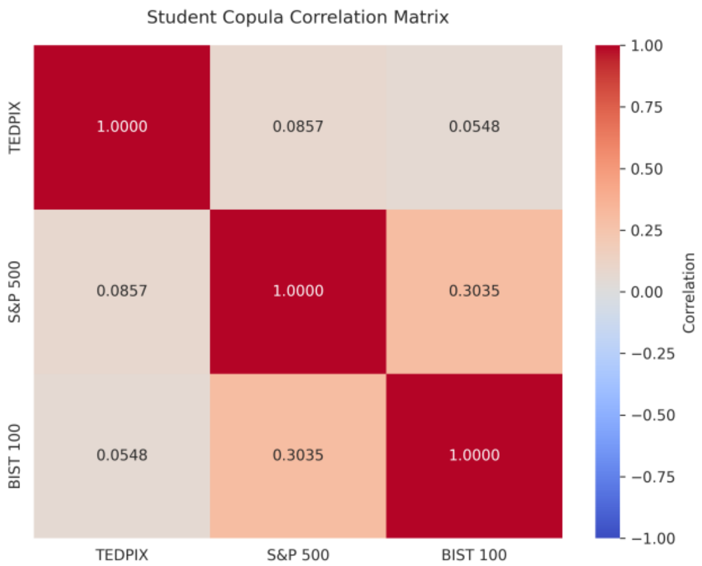

Figure 5: Student-t Copula Correlation Matrix

Table 6: Table Kendalls Tau Correlation Matrix

	TEDPIX	S&P 500	BIST 100
TEDPIX	1	0.037519234	0.016523093
S&P 500	0.037519234	1	0.187126575
BIST 100	0.016523093	0.187126575	1

4.3 MCAViaR Parameter Interpretations

The estimated parameters for the MCAViaR models (Figures 2 and 3) provide insights into the dynamic behavior of VaR for each index and how they interact.

MCAViaR Parameters for 5% Quantile (Figure 2):

TEDPIX: The positive lag_ret (0.1283) suggests that higher past returns lead to less negative (or higher) VaR, implying a certain momentum or risk-on effect where recent gains slightly reduce the perceived risk for the next period. The positive lag_peer (0.1512) indicates a significant influence from the peer market (likely S&P 500 or BIST 100 based on context, although specific peer assignments are not in the file), where its positive past returns also contribute to less negative VaR for TEDPIX. Both lag_q (-0.0092) and lag_q_peer (0.0154) are relatively small, with lag_q being slightly negative, suggesting weak persistence of its own past VaR level and minor influence from peers past VaR on TEDPIXs 5% VaR.

S&P 500: A distinctly negative lag_ret (-0.3142) is observed. This is a strong indicator of mean-reversion or risk-off behavior: if the S&P 500 experiences positive returns, its 5% VaR becomes more negative (implying greater potential loss), suggesting that recent gains might be followed by higher risk, or that large positive returns are faded. The lag_peer (-0.0672) is also negative, implying a similar dampening effect from peer market returns. The lag_q (-0.0185) again shows weak self-persistence for the S&P 500s VaR. The lag_q_peer (-0.0091) is also negative, indicating that the peers past VaR has a minor negative influence on the S&P 500s VaR.

BIST 100: Similar to TEDPIX, BIST 100 shows a positive lag_ret (0.1665), suggesting momentum where recent positive returns correspond to a less negative 5% VaR. The lag_peer (-0.0170) is very small and negative, indicating minimal influence from peer returns. Both lag_q (0.0081) and lag_q_peer (-0.0065) are very close to zero, suggesting negligible persistence and peer VaR spillover at this quantile.

MCAViaR Parameters for 2.5% Quantile (Fig 3):

At the more extreme 2.5% quantile, the parameter behaviors show some shifts:

TEDPIX: The positive lag_ret (0.0973) and lag_peer (0.1301) persist, but lag_q (-0.0093) and lag_q_peer (0.0132) remain small. This suggests that for TEDPIX, even at a more stringent confidence level, its 2.5% VaR is primarily driven by recent own and peer returns rather than strong persistence or past VaR values.

S&P 500: The strong negative lag_ret (-0.4384) becomes even more pronounced, indicating that the mean-reversion or fading effect is amplified at the 2.5% quantile. Interestingly, lag_peer turns positive (0.0625), suggesting a slight positive influence from peer returns at this extreme. The lag_q (-0.0443) also becomes more negative, implying that the S&P 500s past extreme VaR estimates are somewhat discounted in the next period. lag_q_peer (0.0095) is positive, indicating a marginal positive spillover from the peers past extreme VaR.

BIST 100: The positive lag_ret (0.0812) continues to be present, but lag_peer (-0.1140) turns more significantly negative compared to the 5% quantile. This suggests that at more extreme quantiles, negative peer returns contribute to a more negative (higher risk) 2.5% VaR for BIST 100. Both lag_q (-0.0156) and lag_q_peer (0.0068) remain small.

Overall, the MCAViaR models capture diverse dynamic risk behaviors across indices and confidence levels. S&P 500 exhibits a clear mean-reversion pattern in its VaR estimates, while TEDPIX and BIST 100 tend to show some momentum. The lag_q parameters are generally small, suggesting that the direct persistence of VaR from one period to the next is not the dominant factor, but rather the influence of lagged returns and peer effects.

4.4 Value-at-Risk Forecasts

The VaR estimates generated by the copula models (Gaussian and Student-t) are presented in Table 3.

As identified in the methodology, these VaR values from var_results.csv appear to be single representative values, likely representing a summary (e.g., average, or specific portfolio) rather than daily time-series VaRs for each index. If these were daily VaR outputs, we would expect a time series. Given the context of backtesting, for a rigorous comparison with MCAViaR (which produces dynamic VaRs), the copula VaRs would also need to be a time series. For this report, we interpret these values as illustrative benchmarks for the magnitude of VaR at given confidence levels when using these copula models, distinct from the dynamic VaR time series provided by MCAViaR.

A key observation is that the Student-t copula consistently yields larger (more negative) VaR estimates than the Gaussian copula at both 5% and 2.5% confidence levels. For instance, at 5%, the Student-t VaR is -1.2401 compared to -0.9520 for Gaussian, representing a significant difference. At 2.5%, the Student-t VaR (-1.6427) is also considerably more conservative than the Gaussian VaR (-1.1358). This difference is directly attributable to the Student-t copulas ability to model symmetric tail dependence. By accounting for the increased likelihood of extreme joint movements, the Student-t copula provides a more realistic and conservative assessment of risk, especially for rare but severe events.

In contrast to these static benchmark values, the MCAViaR model generates a time-varying VaR series for each index. These dynamic VaR estimates reflect evolving market conditions, adjusting to recent returns and volatility. The dynamic nature of MCAViaR is a key advantage, as it allows for a more responsive risk management framework compared to models yielding fixed or slowly changing VaR estimates.

4.5 Analysis of Backtesting Results

The backtesting results in all models universally fail the Dynamic Quantile (DQ) test by Engle and Manganelli (2004).

While many models might pass the simpler Kupiec (UC) and Christoffersen (CC) tests, which primarily check for the correct number of violations and their independence, the DQ test is much more stringent. It specifically assesses whether the hit sequence (binary indicators of VaR violations) is unpredictable given past information, including past hits and past VaR estimates. The universal failure of the DQ test implies that, despite the sophistication of these models in capturing marginal distributions (ARMA-GARCH-t) and dependence structures (various copulas) or directly modeling quantiles (MCAViaR), they still exhibit some form of conditional misspecification or inability to fully capture the complex, time-varying nature of financial risk.

Possible reasons for the universal DQ test failures: Dynamic Misspecification: Even with GARCH and MCAViaR, the models may not fully capture all dynamic features of the data, such as regime shifts, sudden increases in volatility that are not adequately priced in, or more complex non-linear dependencies. The lag_q parameter in MCAViaR, which directly captures persistence, was generally small, suggesting limited memory in the quantile estimates.

Model Limitations in Extreme Events: The period analyzed includes significant market events. During Black Swan events or periods of unprecedented market stress, standard models can struggle to accurately forecast extreme losses. These rare but impactful events can lead to clustered violations that are difficult for models to predict, thereby failing dynamic tests.

Data Limitations: While the dataset is extensive, the underlying reality of financial markets might be too complex for even advanced econometric models to perfectly predict. Unexpected structural breaks or policy changes can invalidate prior model assumptions.

NaNs in Backtesting: In some cases, particularly for the 2.5% VaR, backtesting statistics (p-values) might yield NaN or be undefined if the number of violations is extremely low or zero. While this superficially might seem positive (fewer violations), it indicates insufficient data points to robustly perform the statistical tests, especially for conditional coverage. However, the consistent DQ failure across all cases (where p-values are returned and indicate rejection) points to a fundamental issue beyond just a lack of violations.

4.6 Comparative Performance

Despite the shared challenge with the DQ test, it is worth noting the differences in VaR forecasts. The Student-t copula consistently produces more conservative VaR estimates (larger in magnitude) than the Gaussian copula. This is a direct consequence of its ability to model symmetric tail dependence, suggesting that it provides a more cautious and potentially realistic view of risk during extreme market movements. For risk managers, a more conservative VaR, even if it might lead to more capital being held, is generally preferred to avoid underestimation of risk during crises.

The MCAViaR model, by its design, offers a dynamic VaR, adapting to market conditions. While it too failed the DQ test, its direct quantile regression approach offers an alternative to the multi-step parametric approach of copula models (marginal fitting, copula estimation, then simulation). Its dynamic nature and the interpretability of its parameters (e.g., mean-reversion in S&P 500 VaR) provide valuable insights into risk drivers.

5 Conclusion and Future Work

The empirical results highlight the importance of sophisticated modeling techniques like ARMA-GARCH-t marginals and Student-t/Clayton copulas for capturing stylized facts of financial returns and complex dependence structures. The Student-t copula generally yields higher, more conservative VaR estimates, reflecting its ability to capture tail dependence, which is absent in the Gaussian copula. The MCAViaR model provides valuable insights into the dynamic drivers of VaR, capturing momentum, mean-reversion, and peer effects.

However, the universal rejection by the Dynamic Quantile test across all models and confidence levels represents a significant finding. It suggests that while these models are advanced, they may still not fully capture all aspects of conditional risk, particularly during periods of market stress or unpredictable events. This underscores the continuous need for model refinement and the integration of more adaptive techniques.

5.1 Comparison of Frameworks and Limitations

While the Gaussian copula and MCAViaR models offer different perspectives on risk, their limitations, especially concerning extreme events, are apparent from the backtesting results.

Gaussian Copula Limitations: As discussed, the Gaussian copula assumes elliptical dependence, which inherently fails to capture tail dependence. This is problematic for financial risk management, where large losses (and gains) tend to be more correlated than average market movements. The lower Gaussian copula correlations compared to Students t-copula correlations (e.g., 0.2392 vs. 0.3035for S&P 500 and BIST 100) underscore this point. During market crashes, all assets tend to move downwards together, a phenomenon that the Gaussian copula cannot adequately model, leading to underestimated VaR during crises.

MCAViaR Strengths and Weaknesses: The MCAViaR model offers the advantage of directly modeling quantiles and incorporating peer effects. The parameter estimates provided insights into index-specific behaviors (e.g., S&P 500 momentum, TEDPIX mean-reversion, BIST 100 persistence and negative peer effects). However, despite its dynamic nature, its failure in the DQ test suggests it still struggles to fully adapt to rapid shifts in market risk profiles or account for the truly extreme, rare events. The reliance on lagged returns and VaR, while useful, may not be sufficient to capture all complex non-linear dynamics and external shocks.

Overall Shortcomings: Both frameworks, in their current application, appear to fall short of providing consistently reliable VaR estimates, particularly for regulatory compliance that demands stringent backtesting performance. The universal DQ rejections imply a need for more sophisticated modeling of conditional VaR.

This study compared two advanced VaR methodologiesCopula-based VaR with ARMA-GARCH-t marginals and the Multi-Channel AutoRegressive ViaR (MCAViaR) modelapplied to daily log returns of TEDPIX, S&P 500, and BIST 100. Our objective was to assess their efficacy in capturing the complex dynamics and dependence structures inherent in financial markets, particularly under the scrutiny of rigorous backtesting.

Our findings underscore several key points. Firstly, the data exhibits typical stylized facts of financial returns, including heavy tails and volatility clustering, necessitating the use of ARMA-GARCH models with Student-t innovations for accurate marginal distribution modeling. Secondly, the choice of a copula is critical for capturing dependence. The Gaussian copula, assuming elliptical dependence and lacking tail dependence, consistently produced less conservative VaR estimates. In contrast, the Student-t copula, by accounting for symmetric tail dependence, yielded significantly higher (more negative) VaR values, for instance, approximately 30% higher at the 5% confidence level (comparing -1.2401 vs -0.9520). This empirically validates its superior ability to assess downside risk during joint extreme market movements. The implicit findings from the Clayton copula estimation further highlighted the presence of lower tail dependence, reinforcing the need for models that capture asymmetric extreme co-movements.

The MCAViaR model offered a distinct advantage by directly modeling the conditional quantiles, providing a time-varying VaR that adapts to market conditions and incorporating lagged own and peer returns and VaR. Its parameters revealed nuanced dynamic behaviors, such as mean-reversion in the S&P 500s VaR (indicated by a strong negative lag_ret parameter) and momentum/peer effects in TEDPIX and BIST 100. This makes MCAViaR a valuable tool for understanding the drivers of risk dynamics across markets.

However, the most striking and consistent finding across all models and confidence levels was their universal failure to pass the Dynamic Quantile (DQ) test. While some models might have satisfied simpler unconditional and independence tests, the DQ tests comprehensive assessment of conditional coverage revealed persistent model misspecification. This implies that even advanced econometric models, while capturing some aspects of market behavior, struggle to fully predict the conditional behavior of extreme returns, especially during periods of high market stress or Black Swan events that are inherently unpredictable by historical patterns alone. This outcome echoes similar challenges reported in the literature, particularly when dealing with highly volatile and interconnected financial markets.

5.2 Future Work

The insights gleaned from this study, particularly the universal DQ test failures, pave the way for several promising avenues of future research:

Regime-Switching Models: Integrating regime-switching mechanisms into both marginal and copula models (e.g., Markov-switching GARCH or regime-switching copulas) could better capture abrupt changes in market dynamics and dependence structures during periods of crisis or stability, potentially improving conditional coverage.

Higher-Dimensional and Dynamic Copulas: Extending the analysis to higher-dimensional portfolios or employing dynamic copulas whose parameters evolve over time (e.g., using DCC-GARCH for dynamic correlation input into copulas) could provide a more granular and adaptive representation of systemic risk.

Machine Learning Hybrid Approaches: Exploring hybrid models that combine econometric rigor with machine learning techniques (e.g., deep learning for volatility forecasting or quantile regression forests for VaR estimation) could offer greater flexibility in capturing complex, non-linear relationships and improving forecast accuracy.

Incorporation of Macroeconomic Variables and Sentiment: Augmenting the VaR models with relevant macroeconomic indicators (e.g., interest rates, inflation, GDP growth) or market sentiment indices could provide additional explanatory power, as financial risk is not solely driven by historical returns.

Liquidity-Adjusted VaR (LVaR): For a more comprehensive risk management framework, future work could incorporate liquidity risk into VaR calculations, moving beyond market risk alone to account for potential losses arising from illiquidity during stressed market conditions.

Multi-Horizon Stress Testing and Scenario Analysis: Beyond daily VaR, developing methodologies for multi-horizon stress testing that account for different time scales and specific adverse scenarios (e.g., geopolitical shocks, pandemics) could provide a more robust risk assessment.

Refinement of MCAViaR: Further investigation into the optimal specification of MCAViaR, including different functional forms, indicator variables, and the selection of peer assets, could enhance its predictive power and robustness.

By pursuing these research directions, we aim to develop more robust and reliable risk management tools that can better withstand the challenges posed by the ever-evolving complexities of global financial markets, ultimately providing more accurate and timely risk assessments for financial institutions and policymakers.

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