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Maximum Principle for McKean-Vlasov Dynamic Using Lions Partial-Derivatives with Respect to Probability with Application to Finance

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Abstract:
In this paper, we study optimal control problem for stochastic systems generated by general McKean-Vlasov dynamics. The coefficients of the McKean-Vlasov dynamic depend on the state of the solution process as well as of its probability law. The information available to the controller is possibly less than the whole information. Our necessary maximum principle is established by applying Girsanov's Lemma and Lions's partial-derivatives with respect to probability law. As an application to finance, the conditional random mean-variance portfolio selection problem of McKean-Vlasov type is discussed to illustrate our theoretical results, where the optimal partially observed portfolio has been derived explicitly.

Keywords: McKean-Vlasov stochastic systems; Stochastic optimal control; Probability and stochastic process; Markowitz portfolio theory; Girsanov Lemma; Finance models.
AMS 2000 subject Classification: 60H30; 93E20.

1 Introduction

McKean-Vlasov dynamics are Itô's stochastic systems, where the coefficients of the state equation depend on the time variable, the state of the solution process as well as of its probability law. Buckdahn et al. [3] established the necessary conditions for general mean-field systems, in which the coefficients depend on the state of the solution process as well as of its probability law. Stochastic maximum principles for general mean-field models were later studied in [9, 20, 22].

The modern theory of portfolio is mainly based on Markowitz's portfolio theory introduced in 1952. Since the pioneering works of Markowitz [19], mean-variance portfolio selection model has been widely used in both theoretical and empirical studies, which minimizes the investment risk under certain return level or maxi-

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mizes the investment return under certain risk level. The mean-variance portfolio selection models have paved the foundation for modern portfolio theory and has been widely applied in financial economics, see [4, 5, 8]. Since the development of nonlinear filtering theory, stochastic control problems under partial-observation have received much attention and became a powerful tool in many fields with important applications, such as finance and economics, etc. In many situations, the states of the systems cannot be completely observed; however, some other processes related to the unobservable states can be observed. Such subjects have been discussed by many authors, such as Wang et al. [23], Wang et al. [24], Bensoussan and Yam [2], Wang et al. [25], Miloudi et al. [20], Abada et al. [1], Korichi et al. [13], Korichi and Hafayed [14].

In this paper, we consider the conditional mean-variance portfolio selection problem in incomplete market, where the state process is partially observed via a noisy process. We derive the characterizations of partially observed optimal portfolio control problem of mean-field type. The coefficients of the dynamic depend on the state of the solution process as well as of its law. Our model of partially observed control problem plays an important role in different fields of economics and finance, as conditional mean variance portfolio selection problem with discrete movement in incomplete market and the optimal consumption and portfolio problem under proportional transaction costs, see.Korichi et al. [13], Abada et al. [1] and Pham [22]. Moreover, the exchange rate under uncertainty, where the government has two means of influencing the foreign exchange-rate of its own currency: the government can choose the domestic interest rate, at all times t and the government can intervene in the foreign exchange market by selling or buying large amounts of foreign currency at selected times τ_i.

In financial markets three important objectives of interventions: to influence the level of the exchange rate, to dampen exchange rate volatility or supply liquidity to foreign exchange markets; and to influence the amount of foreign reserves. Banks intervene in foreign exchange markets in order to achieve a variety of overall economic objectives, such as controlling inflation, maintaining competitiveness or maintaining financial stability, see [10, 12, 26]. As an illustration, we study the conditional mean-variance portfolio selection problem with interventions under incomplete information.

The rest of the paper is organized as follows. Section 2 begins with a formulation of conditional mean-variance portfolio selection problem with interventions. In section 3 the mathematical modelisation of the financial problem is given. In Sect. 4, we derive the necessary characterizations of the optimal partially observed portfolio. In Sect. 5 the explicit formula of the optimal partially observed portfolio is established. Finally, some discussions with concluding remarks and future developments are presented in the last Section.

2 Conditional Mean-Variance Portfolio Selection Problem

Partial information or incomplete information, means that the information available to the controller is possibly less than the whole information. That is, any admissible control is adapted to a subfiltration $(\mathcal{G}_t)_t$ of $(\mathcal{F}_t)_t$ $t \geq 0$. This kind of problem, which has potential applications in mathematical finance and mathematical economics, arises naturally, because it may fail to obtain an admissible control with full information in real world applications. In this section, we study a conditional mean-variance portfolio selection problem with incomplete information on the market.

As example, foreign exchange interventions are conducted by monetary authorities (Bank or minister of finance) to influence foreign exchange rates by buying and selling currencies in the foreign exchange market. Suppose that we are given an incomplete market consisting of two investment possibilities:
A risk free-security, (bond) where the price $\Gamma_0(t)$ evolves according to the ordinary differential equation

$$\begin{cases} \mathrm{d}\Gamma_0\left(t\right) = \gamma_0(I_t\left(w\right))\Gamma_0\left(t\right)\mathrm{d}t, \ t \in \left[0,T\right], \\ \Gamma_0\left(0\right) > 0, \end{cases} \tag{1}$$

where

- The map $\gamma_0\left(\cdot\right) : [0,T] \to \mathbb{R}_+$ is a locally bounded continuous deterministic function.
- $I_t\left(w\right)$ is the observable factor process with dynamics governed by a Brownian motion $B(\cdot)$, assumed to be non correlated with the Brownian motion $W(\cdot)$.
- The filtration $\mathcal{F}_t^I$ generated by the observable factor process I_t is equal to the filtration $\mathcal{F}_t^B$ generated by $B(\cdot)$.
- Notice that the market is incomplete as the agent cannot trade in the factor process.

A risky security (stock), where the price $\Gamma_1\left(t\right)$ at time t is given by

$$\begin{cases} \mathrm{d}\Gamma_1\left(t\right) = \Gamma_1\left(t\right)\left[\left(\varsigma(I_t\left(w\right)) + \gamma_0(I_t\left(w\right))\right)\mathrm{d}t + \sigma(I_t\left(w\right))\mathrm{d}W(t)\right] + \mathrm{d}\xi(t,w), \\ \Gamma_1\left(0\right) > 0, \end{cases} \tag{2}$$

Now, in order to ensure that $\Gamma_1\left(t\right) > 0$ for all $t \in \left[0,T\right]$, we assume that

- The map $\varsigma(\cdot) : [0,T] \to \mathbb{R}$, is bounded continuous such that $\varsigma(I_t\left(w\right)) \neq 0$.
- The map $\sigma(\cdot) : [0,T] \to \mathbb{R}$ are bounded continuous maps such that $\sigma(I_t\left(w\right)) > 0$ and
$$\varsigma(I_t\left(w\right)) > \gamma_0(I_t\left(w\right)), \forall t \in [0,T].$$

Let $U(0) = u_0 > 0$ be an initial wealth process. The wealth dynamic is a stochastic differential equation defined by

$$\begin{cases} \mathrm{d}U^\pi(t) = \gamma_0(I_t(w))(U^\pi(t) - \xi(t,w))\mathrm{d}t \\ \qquad\qquad + \pi(t)\left[\varsigma(I_t(w))\mathrm{d}t + \sigma(I_t(w))\mathrm{d}W(t)\right] + \mathrm{d}\xi(t,w), \\ U^\pi(0) = u_0. \end{cases} \tag{3}$$

Here

- $\gamma_0(I_t(w))$: is the interest rate.
- $\varsigma(I_t(w))$: is the excess rate of return.
- $\sigma(I_t(w))$: the volatility (or the dispersion) of the stock price with $\sigma(I_t(w)) \geq \varepsilon$ for some $\varepsilon > 0$.are measurable bounded functions of $I_t(w)$.
- The process $\pi(t)$ (the regular control process) represents the amount invested in the stock at time t, when the current wealth is $U^\pi(t)$ and based on the past partially observations $\mathcal{F}_t^B$ of the factor process.
- $\xi(t)$ is the intervention (singular) control.

The objective of the agent is to minimize over investment strategies a cost functional $\Pi(\cdot,\cdot)$ of the form:

$$\Pi(\pi(\cdot),\xi(\cdot)) = \mathbb{E}^\pi\left[\frac{\delta}{2}\mathbb{V}ar_B^\pi\left(U^\pi(T) - \xi(T) \mid B(T)\right) - \mathbb{E}^\pi(U^\pi(T) - \xi(T) \mid B(T))\right], \tag{4}$$

for some $\delta > 0$, and the wealth process $U^\pi(t)$ is controlled by the amount $\pi(t)$. Here $\mathbb{E}^\pi\left(U^\pi(t) \mid B(t)\right)$ is the conditional-expectation of the stochastic process $U^\pi(\cdot)$ with respect to $B(t)$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P}^\pi)$ defined as a random process such that for any $A \in \sigma(B)$:

$$\int_A \mathbb{E}^\pi\left(U^\pi(t) \mid B(t)\right)\mathrm{d}\mathcal{P}^\pi = \int_A U^\pi(t)\mathrm{d}\mathcal{P}^\pi.$$

and $\mathbb{V}ar_B^\pi\left(U^\pi(t)\right)$ is the conditional-variance defined by

$$\mathbb{V}ar_B^\pi\left(U^\pi\right) = \mathbb{E}^\pi(U^{\pi 2}(t) \mid B(T)) - (\mathbb{E}^\pi(U^\pi(t) \mid B(T)))^2.$$

The economic justification for this model is based on the *Von-Neumann-Morgenstern* expected utility results, discussed by Markowitz [19], see [13, 22].

3 Formulation of the Problem

3.1 Preliminaries and Notations

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$ be a complete filtered probability space on which are defined two independent standard one-dimensional Brownian motions $W(\cdot)$ and $Y(\cdot)$. Let $T > 0$

be a fixed terminal time. Let $\mathbb{R}^n$ is a n-dimensional Euclidean space, $\mathbb{R}^{n\times d}$ the collection of $n \times d$ matrices. Let $\mathcal{F}_t^W$, and $\mathcal{F}_t^Y$ be the natural filtration generated by $W(\cdot)$ and $Y(\cdot)$ respectively. We assume that $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^Y \vee \mathcal{N}$, where $\mathcal{N}$ denotes the totality of $\mathcal{P}$-null sets. We denote by $\mathbb{E}(\cdot)$ denotes the expectation on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$. Throughout this work, we denote by $\mathbb{L}^2(\mathcal{F}_t; \mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-measurable random variable X, such that $\mathbb{E}(|X|^2) < +\infty$

Let $\Sigma_2(\mathbb{R}^d)$ be the space of all probability measures μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with finite second moment, i.e, $\int_{\mathbb{R}^d} |x|^2 \mu(\mathrm{d}x) < +\infty$, endowed with the following Wasserstein metric $\rho_2(\cdot,\cdot)$; for $\mu, \nu \in \Sigma_2(\mathbb{R}^d)$,

$$\rho_2(\mu,\nu) = \inf_{\delta(\cdot,\cdot)\in \not{\Sigma}_2(\mathbb{R}^{2d})} \left\{ \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x-y|^2 \delta(\mathrm{d}x, \mathrm{d}y) \right]^{\frac{1}{2}} \right\},$$

where $\delta(\cdot,\cdot) \in \Sigma_2(\mathbb{R}^{2d})$, $\delta(A, \mathbb{R}^d) = \mu(A)$, $\delta(\mathbb{R}^d, B) = v(B)$.

3.2 Lions's Partial Derivatives with Respect to Probability law

Now, we recall briefly the innovative notion of $\mathcal{L}$-partial derivatives with respect to probability distribution over Wasserstein spaces, which was studied by Lions [17], and Cardaliaguet [6] and the pioneering work by Cardaliaguet et. al. [7] in their study of the so-called master equation in mean field game systems. The main idea is to identify a distribution $\mu \in \Sigma_2(\mathbb{R}^d)$ with a random variables $X \in \mathbb{L}^2(\mathcal{F}; \mathbb{R}^d)$ so that $\mu = \mathcal{P}_X$. To be more precise, we assume that probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$ is rich enough in the sense that for every $\mu \in \Sigma_2(\mathbb{R}^d)$, there is a random variable $X \in \mathbb{L}^2(\mathcal{F}; \mathbb{R}^d)$ such that $\mu = \mathcal{P}_X$. We denote $\partial_\mu f(\mathcal{P}_Z, y) = \psi[\mu_0](y)$, $y \in \mathbb{R}^d$. We note that for each $\mu \in \Sigma_2(\mathbb{R}^d)$, $\partial_\mu f(\mathcal{P}_Z, \cdot) = \psi[\mathcal{P}_Z](\cdot)$ is only defined in a $\mathcal{P}_Z(dx) - a.e$ sense, where $\mu = \mathcal{P}_Z$.

3.3 Space of Differentiable Functions in $\Sigma_2(\mathbb{R}^d)$

We say that the function $f \in \mathbb{C}_b^{1,1}(\Sigma_2(\mathbb{R}^d))$ if for all $\vartheta \in \mathbb{L}^2(\mathcal{F}; \mathbb{R}^d)$, there exists a $\mathcal{P}_\vartheta$-modification of $\partial_\mu f(\mathcal{P}_\vartheta, \cdot)$ such that $\partial_\mu f : \Sigma_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded and Lipschitz continuous. That is for some $C > 0$, it holds that: $\partial_\mu f(\mu, x)$ is bounded and lipschitzian in μ and x for any $\mu \in \Sigma_2(\mathbb{R}^d)$, and any $x \in \mathbb{R}^d$.
Let ϑ_1 be a closed-convex subset of $\mathbb{R}^k$ and $\vartheta_2 := [0, +\infty)^n$. An admissible continuous control $\pi(\cdot)$ is an $\mathcal{F}_t^Y$-adapted process with values in ϑ_1 satisfies $\sup_{t\in[0,T]} (\mathbb{E}|\pi(t)|^n) < \infty$, see [10, 12]. We denote by $\mathcal{U}_1^Y$ the set of the admissible regular control variables. An intervention control is a stochastic irregular process $\xi(\cdot)$ of measurable ϑ_2-valued, $\mathcal{F}^Y-$adapted processes, such that the process $\xi(\cdot) : [0,T] \times \Omega \to \vartheta_2$ is non-decreasing continuous on the right with left-limits, with bounded variation and $\xi(0) = 0$. Moreover, $\mathbb{E}(|\xi(T)|^p) < \infty$ for any $p \geq 2$.

We denote by $\mathcal{U}_2^Y$ the set of the admissible irregular (singular) control variables. An admissible combined regular-singular control is a pair $(\pi(\cdot), \xi(\cdot))$ of measurable

$\vartheta_1 \times \vartheta_2-$valued, $\mathcal{F}^Y-$adapted processes, such that the process $\pi(\cdot) : [0,T] \times \Omega \to \vartheta_1$ is regular process satisfies and $\xi(\cdot) : [0,T] \times \Omega \to \vartheta_2$ is singular control. We denote by $\mathcal{U}_1^Y \times \mathcal{U}_2^Y$ the set of the admissible combined control variables.

In this paper, we formulate this problem mathematically as a partially observed optimal stochastic intervention control problem for systems governed by MKSDEs with correlated noisy between the system and the observation of the form

$$\begin{cases} \mathrm{d}x^{\pi,\xi}(t) = f(w,t,x^{\pi,\xi}(t), \mathcal{L}[x^{\pi,\xi}(t)], \pi(t))\mathrm{d}t \\ \qquad + \sigma(w,t,x^{\pi,\xi}(t), \mathcal{L}[x^{\pi,\xi}(t)], \pi(t))\mathrm{d}W(t) \\ \qquad + g(w,t,x^{\pi,\xi}(t), \mathcal{L}[x^{\pi,\xi}(t)], \pi(t))\mathrm{d}\widetilde{W}(t) \\ \qquad + Q(t,w)\mathrm{d}\xi(t), \\ x^{\pi,\xi}(0) = x_0, \end{cases} \tag{5}$$

where $\mathcal{L}[X(t)] = \mathcal{P}\circ(X(t))^{-1}$denotes the probability law of the random variable X. The mappings

$$\begin{aligned} f &: \Omega \times [0,T] \times \mathbb{R}^n \times \Sigma_2(\mathbb{R}^d) \times \vartheta_1 \to \mathbb{R}^n \\ \sigma &: \Omega \times [0,T] \times \mathbb{R}^n \times \Sigma_2(\mathbb{R}^d) \times \vartheta_1 \to \mathcal{M}(\mathbb{R}^{n\times d}) \\ g &: \Omega \times [0,T] \times \mathbb{R}^n \times \Sigma_2(\mathbb{R}^d) \times \vartheta_1 \to \mathcal{M}(\mathbb{R}^{n\times d}) \\ Q &: [0,T] \times \Omega \to \mathbb{R}^n \end{aligned}$$

are given functions. In this paper, we suppose that the state process $x^{u,\xi}(\cdot)$ cannot be observed directly, but the controllers can observe partially a related noisy-process $Y(\cdot)$, which is governed by the following equation.

$$\begin{cases} dY(t) &= h(w,t,x^{\pi,\xi}(t), \pi(t))dt + d\widetilde{W}(t) \\ Y(0) &= 0, \end{cases} \tag{6}$$

where $h : \Omega \times [0,T] \times \mathbb{R}^n \times \vartheta_1 \to \mathbb{R}^r$, and $\widetilde{W}(\cdot)$ is a stochastic process depending on the control $u(\cdot)$.
Consider the cost functional $\Pi(\cdot,\cdot)$:

$$\begin{aligned} \Pi(\pi(\cdot), \xi(\cdot)) &= \mathbb{E}^\pi \int_0^T \ell(w,t,x^{\pi,\xi}(t), \mathcal{L}[x^{\pi,\xi}(t)], \pi(t))\mathrm{d}t + \mathbb{E}^\pi \int_{[0,T]} K(t,w)\mathrm{d}\xi(t) \\ &\quad + \mathbb{E}^\pi \left[\psi(w,x^{\pi,\xi}(T), \mathcal{L}[x^{\pi,\xi}(T)])\right], \end{aligned} \tag{7}$$

where $\ell : \Omega \times [0,T] \times \mathbb{R}^n \times \Sigma_2(\mathbb{R}) \times \vartheta_1 \to \mathbb{R}$, $\psi : \Omega \times \mathbb{R}^n \times \Sigma_2(\mathbb{R}) \to \mathbb{R}$ and $\mathbb{E}^\pi(\cdot)$ is the mathematical expectation on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P}^\pi)$ defined by

$$\mathbb{E}^\pi(X) = \mathbb{E}_{\mathcal{P}^\pi}(X) = \int_\Omega X(w)\mathrm{d}\mathcal{P}^\pi(w).$$

3.4 Conditions

We will always take the following standing assumptions in this paper.
Conditions (C1) The maps $f, \sigma, g, \ell : [0,T] \times \mathbb{R} \times \Sigma_2(\mathbb{R}) \times \vartheta_1 \to \mathbb{R}$ and $\psi : \mathbb{R} \times \Sigma_2(\mathbb{R}) \to \mathbb{R}$ are measurable in all variables. Moreover, $f(t,\cdot,\cdot,\pi)$, $\sigma(t,\cdot,\cdot,\pi)$, $g(t,\cdot,\cdot,\pi)$, $\ell(t,\cdot,\cdot,\pi)$, and $\psi(\cdot,\cdot) \in \mathbb{C}_b^{1,1}(\mathbb{R} \times \Sigma_2(\mathbb{R}), \mathbb{R})$ for all $\pi \in \vartheta_1$.

Conditions (C2) Let $\varphi(w,x,\mu) = f(w,t,x,\mu,\pi)$, $\sigma(w,t,x,\mu,\pi)$, $g(w,t,x,\mu,\pi)$, $\ell(w,t,x,\mu,\pi)$, $\psi(x,\mu)$, the function $\varphi(\cdot,\cdot)$ satisfies the following properties:
(**1**) For fixed $x \in \mathbb{R}$ and $\mu \in \Sigma_2(\mathbb{R})$, the function $\varphi(\cdot,\mu) \in \mathbb{C}_b^1(\mathbb{R})$ and $\varphi(x,\cdot) \in \mathbb{C}_b^{1,1}(\Sigma_2(\mathbb{R}^d), \mathbb{R})$. All the derivatives φ_x and $\partial_\mu \varphi$, for $\varphi = f, \sigma, g, \ell, \psi$ are bounded and Lipschitz continuous, with Lipschitz constants independent of $u \in \vartheta_1$.

(**2**) The functions f, σ, g and ℓ are continuously differentiable with respect to control variable $u(\cdot)$, and all their derivatives are continuous and bounded.

The function h is continuously differentiable in x and continuous in v, its derivatives and h are all uniformly bounded which satisfies the following *Novikov's condition*:

$$\mathbb{E}\left(\exp\left[\frac{1}{2}\int_0^t \left|h(w,s,x^{\pi,\xi}(s),\pi(s))\right|^2 \mathrm{d}s\right]\right) < \infty. \tag{8}$$

Conditions (C3) The functions $Q(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$, and $K(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}^+$ are continuous and bounded.

Clearly, conditions (C3) allows us to define integrals of the form $\int_{[0,T]} Q(t,w)\mathrm{d}\xi(t)$ and $\int_{[0,T]} K(t,w)\mathrm{d}\xi(t)$. Moreover, under conditions (C1), (C2) and (C3), for any $(\pi(\cdot), \xi(\cdot)) \in \mathcal{U}_1^Y \times \mathcal{U}_2^Y$ the MVSDE-(5) admits a unique strong solution $x^{\pi,\xi}(t)$. We define the $\mathcal{F}_t^Y$−martingale $\alpha^\pi(t)$ which is the solution of the equation

$$\begin{cases} \mathrm{d}\alpha^\pi(t) = \alpha^\pi(t) h\left(w,t,x^{\pi,\xi}(t),\pi(t)\right) \mathrm{d}Y(t), \\ \alpha^\pi(0) = 1. \end{cases} \tag{9}$$

This martingale allowed to define a new probability $\mathcal{P}^\pi$ on the space $(\Omega, \mathcal{F})$, to emphasize the fact that it depend on the control $\pi(\cdot)$. It is given by the *Radon-Nikodym derivative*:

$$\left.\frac{\mathrm{d}\mathcal{P}^\pi}{\mathrm{d}\mathcal{P}}\right|_{\mathcal{F}_t^Y} = \alpha^\pi(t). \tag{10}$$

From the linear equation (9), and by a simple computation, we get

$$\alpha^\pi(t) = \exp\left[\int_0^t h(w,s,x^{\pi,\xi}(s),\pi(s))\mathrm{d}Y(s) - \frac{1}{2}\int_0^t \left|h(w,s,x^{\pi,\xi}(s),\pi(s))\right|^2 \mathrm{d}s\right]. \tag{11}$$

This type of equations are called *Doléan-Dade exponential formula.* We note that $\mathbb{E}^\pi(g(X))$ refers to the expected value of $g(X)$ with respect to the probability law

$\mathcal{P}^{\pi}$. Moreover, since $\mathrm{d}\mathcal{P}^{\pi}=\alpha^{\pi}(t)\mathrm{d}\mathcal{P}$, we have

$$\begin{aligned}\mathbb{E}^{\pi}(\varphi(X)) &= \mathbb{E}_{\mathcal{P}^{\pi}}(\varphi(X))=\int_{\Omega}\varphi(X(w))\mathrm{d}\mathcal{P}^{\pi}(w),\\ &= \int_{\Omega}\varphi(X(w))\alpha^{\pi}(t)\mathrm{d}\mathcal{P}(w),\\ &= \mathbb{E}_{\mathcal{P}}(\alpha^{\pi}(t)\varphi(X))=\mathbb{E}\left[\alpha^{\pi}(t)\varphi(X)\right].\end{aligned}$$

By Girsanov's Lemma and conditions (C1), (C2) and (C3), $\mathcal{P}^{\pi}$ is a new probability measure of density $\alpha^{\pi}(t)$. The process

$$\widetilde{W}(t)=Y(t)-\int_0^t h(w,s,x^{\pi,\xi}(s),u(s))\mathrm{d}s,$$

is a standard Brownian motion independent of $B(\cdot)$ and x_0 on the new probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathcal{P}^{\pi})$.

By Radon-Nikodym derivative (10), with the martingale property of $\alpha^{\pi}(t)$, the cost functional (7) can be written as

$$\begin{aligned}\Pi(\pi(\cdot),\xi(\cdot)) &= \mathbb{E}\left[\int_0^T \alpha^{\pi}(t)\ell(w,t,x^{\pi,\xi}(t),\mathcal{L}[x^{\pi,\xi}(t)],\pi(t))\mathrm{d}t\right.\\ &\left.\quad+\alpha^{\pi}(T)\psi(w,x^{\pi,\xi}(T),\mathcal{L}[x^{\pi,\xi}(T)])+\int_{[0,T]}\alpha^{\pi}(t)K(t,w)\mathrm{d}\xi(t)\right].\end{aligned}\tag{12}$$

The main purpose of this paper is to prove stochastic maximum principle in Pontryagin form, also called necessary optimality conditions for the partially observed optimal control of MVSDEs-(5).

4 Characterizations for Optimal Control

In this section, we establish the the explicit characterizations of our partially observed optimal intervention control of general MVSDEs. The proof is based on *Girsanov's lemma*, the Lions partial derivatives with respect to probability law.

Hamiltonian. We define the Hamiltonian

$$H:\Omega\times[0,T]\times\mathbb{R}\times\Sigma_2(\mathbb{R})\times\vartheta_1\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R},$$

associated with our control problem by

$$\begin{aligned}&H(w,t,x,\mu,\pi,p(t),q(t),\overline{q}(t),k(t))\\ =\;&\ell(w,t,x,\mu,\pi)+f(w,t,x,\mu,\pi)p(t)+\sigma(w,t,x,\mu,\pi)q(t)\\ &+g(w,t,x,\mu,\pi)\overline{q}(t)+h(w,t,x,\pi)k(t).\end{aligned}\tag{13}$$

We are now ready to introduce two new adjoint equations that will be the building blocks of the stochastic maximum principle and

$$\begin{cases} -\mathrm{d}p\left(t\right) & =\Big[f_x\left(t\right)p\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu \widehat{f}\left(t\right)\widehat{p}\left(t\right)\right]+\sigma_x\left(t\right)q\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu\widehat{\sigma}\left(t\right)\widehat{q}\left(t\right)\right] \\ & \quad +g_x\left(t\right)\overline{q}\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu\widehat{g}\left(t\right)\widehat{\overline{q}}\left(t\right)\right]+\ell_x\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu\widehat{\ell}\left(t\right)\right] \\ & \quad +h_x\left(t\right)k(t)\Big]\mathrm{d}t-q(t)\mathrm{d}W(t)-\overline{q}(t)\mathrm{d}\widetilde{W}(t) \\ p(T) & =\psi_x(x\left(T\right),\mathcal{P}\left[x(T)\right])+\widehat{\mathbb{E}}\left[\partial_\mu\psi(\widehat{x}\left(T\right),\mathcal{P}\left[x(T)\right])\right]. \end{cases} \tag{14}$$

and

$$\begin{cases} \mathrm{d}y(t) & =z\left(t\right)\mathrm{d}W(t)+k\left(t\right)\mathrm{d}\widetilde{W}\left(t\right)-\ell(t)dt, \\ y(T) & =\psi(x(T),\mathcal{L}\left[x(T)\right]). \end{cases} \tag{15}$$

The main result of this section is stated in the following maximum principle. This approach will be used to solve explicitly our conditional mean-variance portfolio selection problem (19)-(20).

Theorem 4.1 Let conditions (C1), (C2) and (C3) hold. Let $\left(\overline{\pi}(\cdot),\overline{\xi}(t)\right)$ be the optimal solution of the control problem (5)-(7). Let H be the Hamiltonian function defined by (13). Then there exists $\mathcal{F}_t^Y-$adapted adjoint processes $\left(p\left(\cdot\right),\ q\left(\cdot\right),\overline{q}(\cdot),k\left(\cdot\right)\right)$ solution of (14)-(15) such that for any $(\pi,\xi)\in\vartheta_1\times\vartheta_2$, we have $\mathbb{P}-a.s.,\ a.e.t\in[0,T]$.

$$\begin{aligned} 0 \quad \leq \quad & \mathbb{E}^\pi\left[\partial_\pi H(t,\overline{x}(t),\mathcal{L}\left[\overline{x}(t)\right],\overline{\pi}\left(t\right),p\left(t\right),q\left(t\right),\overline{q}\left(t\right),k\left(t\right))\left(\pi\left(t\right)-\overline{\pi}\left(t\right)\right)\mid\mathcal{F}_t^Y\right] \\ & +\mathbb{E}^u\left[\int_{[0,T]}(Q(t)+K(t)p(t))\mathrm{d}\left(\xi-\overline{\xi}\right)(t,w)\mid\mathcal{F}_t^Y\right]. \end{aligned} \tag{16}$$

Proof. Applying the similar arguments developed in Korichi et al. [13, Theorem 3.1, with $g=0$.]. □

5 Optimal Partially Observed Portfolio

In this section, we study a conditional mean-variance portfolio selection problem in incomplete market. We establish the optimal partially observed portfilio explicitly. We give some examples. If $f\left(\mu\right)=\int_{\mathbb{R}^n}\varphi(x)\mu(\mathrm{d}x)$, where $\mu(\mathrm{d}x)=\frac{1}{x}\exp(-x)I_{\{x>0\}}\mathrm{d}x$ then the $\mathcal{L}$-partial derivatives of $f\left(\mu\right)$ with respect to measure $\mu(\mathrm{d}x)$ at z is given by

$$\partial_\mu f\left(\mu\right)\left(z\right)=\frac{\partial\varphi}{\partial x}\left(z\right). \tag{17}$$

If $f\left(\mu\right)=\int_{\mathbb{R}^n}\varphi(x,\mu)\mu(\mathrm{d}x)$ then the $\mathcal{L}$-partial derivatives of $f\left(\mu\right)$ with respect to measure at z is given by

$$\partial_\mu f\left(\mu\right)\left(z\right)=\frac{\partial\varphi}{\partial x}\left(z,\mu\right)+\int_{\mathbb{R}^n}\frac{\partial\varphi}{\partial\mu}(x,\mu)\left(z\right)\mu(\mathrm{d}x). \tag{18}$$

Optimal partially observed portfolio. In this section, we establish the optimal partially observed portfolio for conditional mean-variance portfolio selection problem in incomplete market with interventions. The objective of the agent is to minimize over investment strategies a cost functional of the form:

$$\Pi\left(\pi(\cdot),\xi(\cdot)\right)=\mathbb{E}^{\pi}\left[\frac{\delta}{2}\mathbb{V}ar^{\pi}\left(U(T)-\xi(T)\mid B(T)\right)-\mathbb{E}^{\pi}(U(T)-\xi(T)\mid B(T))\right], \tag{19}$$

for some $\delta>0$, with a dynamics for the wealth process $U^{\pi}(t)$ controlled by the amount $\pi(t)$, subject to

$$\begin{cases} \mathrm{d}U(t)=\gamma_0(I_t(w))(U(t)-\xi(t))\mathrm{d}t+\pi(t)\left[\varsigma(I_t(w))\mathrm{d}t+\sigma(I_t(w))\mathrm{d}W(t)\right]+\mathrm{d}\xi(t,w) \\ U(0)=x_0. \end{cases} \tag{20}$$

Let $z(t)=U(t)-\xi(t)$, then the system-(20) has the form:

$$\begin{cases} \mathrm{d}z(t)=\gamma_0(I_t(w))z(t)\mathrm{d}t+\pi(t)\left[\varsigma(I_t(w))\mathrm{d}t+\sigma(I_t(w))\mathrm{d}W(t)\right], \\ z(0)=x_0. \end{cases} \tag{21}$$

and the cost functional $\Pi\left(\pi(\cdot),\xi(\cdot)\right)$ has the form

$$\Pi\left(\pi(\cdot),\xi(\cdot)\right)=\mathbb{E}^{\pi}\left[\frac{\delta}{2}\mathbb{V}ar^{\pi}\left(z(T)\mid B(T)\right)-\mathbb{E}^{\pi}(z(T)\mid B(T))\right], \tag{22}$$

where $\mathbb{E}^{\pi}(z(t)\mid B(t))$ is the conditional expectation given $B(t)$ and $\mathbb{V}ar^{\pi}\left(z(t)\mid B(t)\right)$ is the conditional variance with respect to $\mathcal{P}^{\pi}$ given $B(t)$. We note that the law of total variance is given by

$$\mathbb{V}ar^{\pi}\left(z(t)\right)=\mathbb{V}ar^{\pi}\left(z(t)\mid B(t)\right)+\mathbb{V}ar^{\pi}\left[\mathbb{E}^{\pi}(z(t)\mid B(t))\right].$$

By applying Theorem 4.1, where $f(w,t,z,\pi)=\gamma_0(I_t(w))z(t)+\pi(t)\varsigma(I_t(w))$, $\sigma(w,t,z,\pi)=\sigma(I_t(w))$, $\ell(w,t,z,\mu,\pi)=h(w,t,z,\mu,\pi)=g(w,t,z,\mu,\pi)=0$ and

$$H(w,t,z,\mu,\pi,p,q,\overline{q},k)=\left[\gamma_0(I_t(w))z(t)+\pi(t)\varsigma(I_t(w))\right]p(t)+\sigma(I_t(w))q(t),$$

then following Hafayed [11] and Pham [22] the optimal control $\overline{\pi}(t)$ of (21)-(22) is given in feedback form

$$\begin{aligned} \overline{\pi}(t) \quad &= \quad \frac{\varsigma(I_t(w))}{\sigma^2(I_t(w))}\left[\mathbb{E}^{\pi}(z^*(t)\mid B(t))-\overline{z}(t)\right] \\ &\quad +\frac{\varsigma(I_t(w))}{\sigma^2(I_t)\varphi_3(t)}\left[\frac{1}{2}\varphi_2(t)-\varphi_1(t)\mathbb{E}^{u}\left(\overline{z}(t)\mid B(t)\right)\right], \end{aligned} \tag{23}$$

where $z(t)$ is given by Eq-(21), and the stochastic processes $\varphi_1(t),\varphi_2(t)$, and $\varphi_3(t)$ satisfy the following SDEs $t\in[0,T]$

$$\begin{aligned} \mathrm{d}\varphi_1(t) \quad &= \quad \left[\frac{\varphi_1(t)^2\varsigma^2(I_t(w))}{\varphi_3(t)\sigma^2(I_t(w))}-2\gamma_0(I_t(w))\varphi_1(t)\right]\mathrm{d}t+Z_t^a\mathrm{d}B(t), \\ \varphi_1(T) \quad &= \quad 0. \end{aligned} \tag{24}$$

$$\mathrm{d}\varphi_2(t)=\left[\frac{\varphi_1(t)\varsigma^2(I_t(w))}{\varphi_3(t)\sigma^2(I_t(w))}-\gamma_0(I_t(w))\right]\mathrm{d}t+Z_t^b\mathrm{d}B(t),\ \varphi_2(T)=-1. \tag{25}$$

and

$$\mathrm{d}\varphi_3(t)=\left[\frac{\varsigma^2(I_t(w))}{\sigma^2(I_t(w))}-2\gamma_0(I_t(w))\right]\varphi_3(t)\mathrm{d}t+Z_t^c\mathrm{d}B(t),\ \varphi_3(T)=\frac{\delta}{2}. \tag{26}$$

The explicit solutions of the above equations are given by

$$\begin{aligned}\varphi_1(t) &\equiv 0,\ \forall t\in[0,T],\\ \varphi_2(t) &= -\mathbb{E}^\pi\left[\exp\int_t^T\gamma_0(I_s(w))\mathrm{d}s\mid\mathcal{F}_t^B\right].\\ \varphi_3(t) &= \frac{\delta}{2}\mathbb{E}^\pi\left[\exp\int_t^T(2\gamma_0(I_s(w))-\frac{\varsigma^2(I_s(w))}{\sigma^2(I_s(w))})\mathrm{d}s\mid\mathcal{F}_t^B\right];\end{aligned} \tag{27}$$

Hence, substituting (27), into (23) yields

$$\begin{aligned}\overline{\pi}(t) &= \frac{\varsigma(I_t(w))}{\sigma^2(I_t)}\left[x_0\exp\left(\int_0^t\gamma_0(I_\tau(w))\mathrm{d}\tau\right)-\overline{z}(t)\right.\\ &\left.+\int_0^t\frac{\varsigma^2(I_t(w))}{2\sigma^2(I_t(w))}\frac{|\varphi_2(s)|}{\varphi_3(s)}\exp\left(\int_0^t\gamma_0(I_\tau(w))\mathrm{d}\tau\right)\mathrm{d}s+\frac{|\varphi_2(t)|}{\varphi_3(t)}\right].\end{aligned} \tag{28}$$

Finally, we deduce that the optimal control of the problem (20)-(19) is given in feedback form

$$\begin{aligned}\overline{\pi}(t) &= \frac{\varsigma(I_t(w))}{\sigma^2(I_t)}\left[x_0\exp\left(\int_0^t\gamma_0(I_s(w))\mathrm{d}s\right)-\overline{U}(t)+\xi(t)\right.\\ &\left.+\int_0^t\frac{\varsigma^2(I_t(w))}{2\sigma^2(I_t(w))}\frac{|\varphi_2(s)|}{\varphi_3(s)}\exp\left(\int_0^t\gamma_0(I_\tau(w))\mathrm{d}\tau\right)\mathrm{d}s+\frac{|\varphi_2(t)|}{\varphi_3(t)}\right].\end{aligned} \tag{29}$$

Now, let $\overline{\xi}(t)$ be $\mathcal{F}_t^Y-$adapted process satisfies Theorem 4.1, then for any $\xi(\cdot)\in\mathcal{U}_2^Y$ we get

$$\begin{aligned}&\mathbb{E}^\pi\left[\int_{[0,T]}(K(t)+Q(t)p(t))\mathrm{d}\overline{\xi}(t,w)\mid\mathcal{F}_t^Y\right]\\ \leq\ &\mathbb{E}^\pi\left[\int_{[0,T]}(K(t)+Q(t)p(t))\mathrm{d}\xi(t,w)\mid\mathcal{F}_t^Y\right].\end{aligned} \tag{30}$$

We define a subset $\mathcal{E}\subset[0,T]\times\Omega$ such that

$$\mathcal{E}=\{(t,w)\in[0,T]\times\Omega:K(t)+Q(t)p(t)>0\}, \tag{31}$$

and let $\xi(\cdot)\in\mathcal{U}_2^Y$ defined by

$$\mathrm{d}\xi(t,w)=\begin{cases}0:\text{if }\ (t,w)\in\mathcal{E},\\ \mathrm{d}\overline{\xi}(t,w):\text{if }\ (t,w)\in\overline{\mathcal{E}},\end{cases} \tag{32}$$

where $\overline{\mathcal{E}}$ is the complement of the set $\mathcal{E}$. For any set $\mathcal{E}$, the indicator function of $\mathcal{E}$ is defined by

$$I_{\mathcal{E}}(w) = \begin{cases} 1 \text{ if } w \in \mathcal{E} \\ 0, \ otherwise. \end{cases}$$

From (30) we get

$$\mathbb{E}^{\pi}\left[\int_0^T (K(t) + Q(t)p(t))\mathrm{d}\left(\xi(t,w) - \overline{\xi}(t,w)\right) \mid \mathcal{F}_t^Y\right] \geq 0.$$

By applying (31), the fact that $\mathcal{E}\subset\Omega \times [0,T]$ and $\xi(t,w)|_{\mathcal{E}} = 0$, we obtain

$$\begin{aligned} 0 &\leq \mathbb{E}^{\pi}\left[\int_0^T (K(t) + Q(t)p(t))I_{\mathcal{E}}(t,w)\mathrm{d}\left(-\overline{\xi}\right)(t,w) \mid \mathcal{F}_t^Y\right] \\ &\quad + \mathbb{E}^{\pi}\left[\int_0^T (K(t) + Q(t)p(t))I_{\overline{\mathcal{E}}}(t,w)\mathrm{d}\left(\overline{\xi} - \overline{\xi}\right)(t,w) \mid \mathcal{F}_t^Y\right] \\ &= -\mathbb{E}^{\pi}\left[\int_0^T (K(t) + Q(t)p(t))I_{\mathcal{E}}(t,w)\mathrm{d}\overline{\xi}(t,w) \mid \mathcal{F}_t^Y\right], \end{aligned}$$

this implies that $\overline{\xi}(\cdot)$ satisfies for any $t \in [0,T]$:

$$\mathbb{E}^{\pi}\left[\int_0^T (K(t) + Q(t)p(t))I_{\mathcal{E}}(t,w)\mathrm{d}\overline{\xi}(t,w) \mid \mathcal{F}_t^Y\right] = 0.$$

From (31) and (32), we can easy shows that the optimal singular control has the form

$$\overline{\xi}(t,w) = \int_0^t I_{\overline{\mathcal{E}}}(s,w)\mathrm{d}s + \xi(t,w), \ t \in [0,T].$$

Finally, as a summary, we give the explicit optimal partially observed portfolio in feedback form in the following theorem

Theorem 5.1. The optimal partially observed portfolio for conditional mean-variance portfolio selection problem (19)-(20).is given in the following feedback

$$\begin{aligned} \overline{\pi}(t,\overline{U}(t)) &= \frac{\varsigma(I_t(w))}{\sigma^2(I_t(w))}\left[x_0 \exp\left(\int_0^t \gamma_0(I_s(w))\mathrm{d}s\right) - \overline{U}(t) + \overline{\xi}(t,w)\right. \\ &\quad \left. + \int_0^t \frac{|\varphi_2(s)|}{2\varphi_3(s)}\frac{\varsigma^2(I_t(w))}{\sigma^2(I_t(w))}\exp\left(\int_0^t \gamma_0(I_\tau(w))\mathrm{d}\tau\right)\mathrm{d}s + \frac{|\varphi_2(t)|}{\varphi_3(t)}\right], \end{aligned}$$

where $\varphi_2(t)$ and $\varphi_3(t)$ are given by (27).

6 Conclusion

In this paper, maximum principle for MVSDEs with noisy observations has been established. Optimal partially observed portfolio of conditional mean-variance portfolio selection problem with interventions is studied. The main result has been established by applying Girsanov's Lemma and the Lions's partial derivatives, where the optimal partially observed portfolio $\overline{\pi}$ is depend to optimal state process and given in feedback form. We hope that in the near future, we can treat other financial problems governed by McKean-Vlasov Backward SDEs.

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