Journal of Mathematics and Modeling in Finance (JMMF)
Vol. 1, No. 1, Winter & Spring 2021
Research paper

Mathematical modeling of stock price behavior and option valuation

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Abstract:
This study emphasizes on the mathematical modeling procedure of stock price behavior and option valuation in order to highlight the role and importance of advanced mathematics and subsequently computer software in financial analysis. To this end, following price process modeling and explaining the procedure of option pricing based on it, the resulting model is solved using advanced numerical methods and is executed by MATLAB software. As derivatives pricing models are based on price behavior of underling assets and are subject to change as a result of variation in the behavior of the asset, studying the price behavior of underlying asset is of significant importance. A number of such models (such as Geometric Brownian Motion and jump-diffusion model) are, therefore, analyzed in this article, and results of their execution based on real data from Tehran Stock Exchange total index are presented by parameter estimation and simulation methods and also by using numerical methods.

Keywords: Stochastic Differential Equations, Stocks, Options, Finite Difference, Monte Carlo Simulation.

1 Introduction

A topic of interest to financial analysts and economists is the valuation and analysis of securities price behavior. In modern financial science, however, accurate analysis of securities price behavior without a quantitative model is not possible. On the other hand, in the field of risk management, risk factors such as price volatility can also be identified and controlled using quantitative models (Brownlees and Galo, 2009). Even in other related fields to finance, such as accounting and auditing we can find many quantitative models (Laura-Diana. (2009)). Fortunately, current financial science researchers have acknowledged the importance of quantitative models to the extent that a great volume of research has been conducted in this area in universities and reputable investment companies, and several books have been published in this area.

One of the most important applications of mathematical modeling of financial variables which can also be localized in Iran is modeling spot and derivatives mar-

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Received: 2020-11-13 Approved: 2020-12-15
http://dx.doi.org/10.22054/jmmf.2020.56846.1022

kets such as oil spot prices or futures (such as Karimnejad Esfahani et al (2020)), short interest rates (Peymany and Hooshangi, 2017) and of course new market of stock options in Tehran stock exchange. Accordingly, the main object of this article is highlighting the procedure of modeling asset price behavior and solving option pricing equations. As the price of each option depends on the value of its underlying asset (such as a particular stock), the underlying assets price behavior is the main determinant of the option value. Therefore, at the first stage, the process of stock price behavior modelling is explained, thereafter, option valuation method is elaborated. Finally, the numerical solution of the valuation model is described. In any section, if necessary, the procedures of executing the introduced items is explained using MATLAB software. Thus, the following sections first introduce the principles of mathematical modeling, types of quantitative models and their application in financial sciences and next, Geometric Brownian Motion is discussed separately as one of the most widely used stochastic quantitative models in financial sciences. Due to certain shortcomings of the model, jump-diffusion models are also described as new practical models in this field. Furthermore, the parameters of both models are estimated based on real data from total index of Tehran Stock Exchange, and the index is simulated based on the data. Additionally, the application of quantitative models in option pricing are explicated. Finally, a general numerical solution is provided to solve option pricing models, and the step-by-step implementation algorithm of the solution is explained.

2 Quantitative Financial Modeling and the Required Tools

Advanced mathematical, financial and economic methods are essential for modeling and solving financial problems, and mathematical, statistical and computer methods required in this field are based on high-level of mathematical knowledge. The basis of modeling a financial or economic quantity is to first examine a financial variable (for example, pricing an option bond), and identify the factors influencing that quantity (such as interest rate, underlying asset price, etc.). Next, by performing mathematical and statistical calculations, it is necessary to locate the desired quantity in one or multiple mathematical equations which is referred to as a model. Finally, using mathematical and computer methods, the designed model is solved and applied. For example, fixed income securities, stocks and derivatives prices can be modeled using different methods such as ordinary or partial differential equations

An option bond is a type of derivative, and derivatives are securities whose value is derived from another underlying asset such as stock. Call (put) option bonds refer to those which give the holder the right to purchase (sell) a certain amount of underlying asset (such as the stock of a particular company) at a specified price (strike price) and a specified time (maturity). There are many types of option bonds, two of which are European and American bonds. A European option may be exercised only at the expiration date of the option while an American option may be exercised at any time before the expiration date (Hull, 2018).

in deterministic or stochastic environments. It is, therefore, essential for a financial researcher to become familiar with certain concepts as the main tools of modeling in order to perform financial quantitative modeling. These tools can be divided into three categories. First, there are financial concepts such as risk, interest rate, securities, portfolios, options, etc., which no comprehensive model can be designed without understanding them. The second category comprises topics in mathematics and statistics such as ordinary differential equations, integral equations, partial differential equations, stochastic differential equations, stochastic partial differential equations, and stochastic processes that provide quantitative account of financial concepts. Finally, the third category includes quantitative modeling tools, namely mathematical and computer methods which are required to implement the designed model. Among this type of tools, numerical methods are the most popular ones as in most financial models, due to the presence of a stochastic sentence, no complete analytical answer can be found and, therefore, numerical methods (such as the finite difference method and Monte Carlo simulation method, etc.) are mostly employed to solve such models. Because heavy and lengthy calculations are required to perform numerical methods, it is impossible to perform them manually, thus it is necessary to use a computer programming language.

Although there is numerous software in this field, one of the most widely used ones is MATLAB software. This is due to the fact that the software, in addition to enjoying a two-way practical programming environment, has a powerful and specialized toolkit (such as partial differential equation toolbox, statistical toolbox, optimization toolbox, and toolboxes specific to financial calculations and derivatives) for solving such problems. The software, therefore, is used to perform calculations in this research.

3 Stock Price Behavior Modeling

As mentioned earlier, price of derivatives depends on the price of the underlying asset. (Hull, 2018). Meanwhile, of all various securities in Iran capital market, ordinary stocks are more popular than other securities. Therefore, stock is focused on in this article as the underlying asset (and the total index of Tehran stock exchange which is the general estimation of stock price volatility is selected as its general index). The following parts will focus on this type securities, and it should be noted that the main topics discussed in the following sections can be easily generalized to other types of securities.

To explicate the modeling procedure of financial variables, first non- stochastic variables and ordinary differential equations are used, and then stochastic variables such as stocks are addressed. Meanwhile, of all quantitative models, the stochastic model of Geometric Brownian Motion is introduced to explain stock price behavior (due to its popularity after being applied in the Black and Scholes (1973) and Merton (1973) models), and in addition to explaining it, the model is solved through a practical example using numerical methods and MATLAB software. Also, by combining this model with Compound Poisson process, a stochastic model including a jump sentence is suggested for pricing high-risk stock that eliminates some of the inefficiencies of the Brownian Motion model. To achieve these goals, mathematical and statistical basics should be introduced which will be described in each section.

3.1 Ordinary Differential Equations in Financial Sciences

If $y = y(t)$ is a single-variable function, then any equation with $y\,(t)$, t and $y\,(t)$ derivatives is called an ordinary differential equation. Most desired differential equations in financial modeling are in the form of $dy/dt \ = ay + f$.Here, if f and a functions are in terms of $y\,(t)$ with its derivatives, then the equation is nonlinear and has generally no analytical answer. To solve the problem, numerical methods should be applied, but if the functions are in t terms, it is proved that the equation has the following answer (Birkohff, 1988):

$$y\,(t) = y_0 e^{A(t)} + \int_0^t e^{A(t)-A(s)} f(s) ds$$

where $A\,(t) = \int_0^t a(\eta)d\eta$. This equation can also be solved using MATLAB software as follows:

```
Gray ■dsolve(Dy=ay+f,y)
```

Among the financial variables that can be modeled using ordinary differential equations, bond price is noteworthy. For example, suppose $B\,(t)$ and $k\,(t)$ are bond price and its risk-free coupon rate (interest), respectively. The final condition is also given at time T of the bond with $B\,(T) = P$ where P is the par value of the bond. At the infinitesimal time of dt from the present time of t, change in the value of the bonds is $\frac{dB}{dt}dt$, and the received coupon is $k\,(t)\,dt$. In the absence of arbitrage, the sum of the above equation should be equal to the risk-free interest rate of r (t) B (t) dt. That is, for $t < T$ the equation is as follows:

$$\frac{dB}{dt}dt + k(t)dt = r(t)Bdt,$$

and by dividing it into dt, the following will be obtained:

$$\frac{dB}{dt} + k(t) = r(t)B, \quad t < T$$

The above equation is a linear equation with the following answer:

$$B(t) = \mathrm{e}^{-\int_t^T r(s)ds} \left[P + \int_t^T k(u) \mathrm{e}^{\int_u^T r(s)ds} du \right]$$

3.2 Stochastic Differential Equations in Financial Sciences

An equation whose coefficients, data or conditions are stochastic or under the influence of an external stochastic factor is called a stochastic equation. Stochastic equations are divided into two groups according to the characteristics of the sample paths of the effective process as follows:

Group 1: Ideal Random Equations

This group of equations are solved through path to path method, and by fixing the path, they become similar to ordinary differential equations. In such equations, the noise sentence is in the form of colored noise, and the system can be converted to the ideal system (noise-free system). This is why this group of equations is called ideal random equations. If there are ordinary derivatives in such equations, the ideal random equation is called the ideal random differential equation. For instance, in the following equation:

$$\frac{\mathrm{ds}}{\mathrm{dt}} = \mathrm{a}\,(\omega)\, s + b\,(t, \omega)$$
$$s_0\,(\omega) = s_0$$

where s is the asset price under study. An example of random differential equation is the ideal linear equation. Now, if ω path in relation to t is constant, the answer to the equation is as follows:

$$\mathrm{x}\,(\mathrm{t}, \omega) = x_0\,(\omega)\, \mathrm{e}^{\mathrm{a}(\omega)\mathrm{t}} + \int_0^{\mathrm{t}} \mathrm{e}^{\mathrm{a}(\omega)(t-\mathrm{s})} \mathrm{b}\,(\mathrm{s}, \omega)\, \mathrm{ds}$$

Where sample paths of the answer are derivative functions in relation to t. Such equations have little application in financial sciences, and therefore, will not be addressed further.

Group 2: Stochastic Differential Equations

Stochastic differential equations are ones in which the response process is not differentiable, and apply when there is a specific, out-of-rule stochastic process in the equation, such as white Gaussian noise (Oksendal, 1998). In stochastic equations, differential and derivative symbols are used symbolically and outside of their conventional meaning, and the integration of these symbolic differentials is carried out through special methods such as Ito or Stratonovich integrals. If the derivative in these equations is ordinary, such equations are called stochastic differential equations. A general example of these equations is as follows:

$$\frac{dS_{\mathrm{t}}}{\mathrm{dt}} = \mathrm{b}\,(\mathrm{t}, S_{\mathrm{t}}) + \sigma\,(t, S_t)\, W_t,$$

where white noise W_t is one-dimensional, and:

$$b\,(t, S_t)\,, \sigma\,(t, S_t) \in \Re$$

The above equation can be converted to the following stochastic differential equation (Oksendal, 1998):

$$dS_{\mathrm{t}} = \mathrm{b}\left(\mathrm{t}, S_{\mathrm{t}}\right)\mathrm{dt} + \sigma\left(\mathrm{t}, S_{\mathrm{t}}\right)\mathrm{dB}_{\mathrm{t}}$$

In fact, to obtain the above equation, white noise W_t is replaced with $\frac{dS_{\mathrm{t}}}{\mathrm{dt}}$ and then the two sides are multiplied by dt. In the equation, $b\left(t, S\right)$ and $\sigma\left(t, S\right)$ functions are called coefficients, and are named as thrust (transfer) and diffusion (volatility) coefficients, respectively.

4 Geometric Brownian Motion Model

A simple example of stochastic differential equations which is also widely used to model stock price is the Geometric Brownian motion. By choosing $b\left(t, S_t\right) = \mu S_t, \sigma\left(t, S_t\right) = \sigma S_t$ we have the following:

$$dS_{\mathrm{t}} = \mu S_{\mathrm{t}}\mathrm{dt} + \sigma \mathrm{S}_{\mathrm{t}}\mathrm{dB}_{\mathrm{t}}$$

which has the following answer using Itos Lemma:

$$S_t = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma z_t}$$

where S_t is the asset price (positive numerical), S_0 is the asset price at time zero, μ is the thrust sentence, σ is the return volatility rate, and $z_{t\geq 0}$ is the standard Brownian motion.

5 Jump-Diffusion Model

A special mode in price behavior procedure of securities is when there are large jumps in price that disrupt the process. This section focuses on explaining models with jump sentence, and describes the mathematical explanation of it. In this group of models, the initial model is completed by adding the jump sentence to the original diffusion equation as follows:

$$dS_t = jump\ + diffusion$$

The mentioned jump sentence is mainly derived from a Poisson distribution (Klugman Willmott, 2005), and in practice too, indicates the greater power of these models in describing the price behavior of risk assets (Andersen and Andersen, 2000). Although the models usually enjoy constant thrust rate and volatility sentence, this condition is not required, and there can be cases in which a nonlinear stochastic differential equation comes along with the jump sentence. Of course, in this case, it is often impossible to obtain an analytical answer for S_t, and therefore, it is necessary to use numerical methods to solve such a stochastic differential equation. One of the most popular jump diffusion models among researchers can be described in general terms as follows:

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t + d(\sum_{i=1}^{N(t)} Y_i)$$

where Y_i is the percentage of jump size with similar independent deterministic distributions, and $N(t)$ is a counting process that in a particular mode, is a Poisson random variable. That is:

$$P\left(N\left(t\right)=n\right) = e^{-\lambda t\frac{(\lambda t)^n}{n!}}$$

$N(t)$ can also be considered as a negative binomial distribution. Here, Y_i is typically a random variable, but in a special mode of it, $Y_i = r$ where r is a constant value. In this paper, it is assumed that Y_i is a random variable i.i.d with a definite distribution.

So far, several advantages of the jump diffusion model have been enumerated. First, the model is able to justify certain important observed characteristics such as skewness or smiling of fluctuations. Second, the jump process can explain the phenomenon of investors failure in the financial market, which means that in real market, it is impossible to react in a short time to avoid loss. The model can also take into account unexpected events (such as market collapse) which play an important role in investors analysis in the real world (Jin-Chuan, et al., 2003and Kou, 2003) or jump diffusion terms in other variables such as interest rate (Mohamadinejad, 2020).

Most of the studied jump diffusion models enjoy acceptable results based on real data. For this type of models, it is assumed that risk assets operate in the form of a linear stochastic differential equation (continuous component) and a Poisson process (jump or discontinuous component). Kou and Wong (2003) for instance, studied a jump-diffusion model for option pricing, based on which it was assumed that the size of jump percentage follows a double exponential function with a probable combined value. As per the results presented by Kou, it can be argued that jump diffusion models can be highly useful for determining option price. By solving this particular model of jump diffusion density, Kou was able to suggest the first-passage time problem for the model.

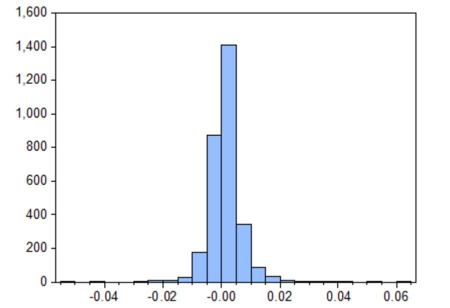
6 Executing Stochastic Models Using Monte Carlo Simulations

In this section, it is attempted to execute the most important items mentioned in the previous sections based on data related to Iran capital market, and present the results. The variable employed is, therefore, the total index of Tehran Stock Exchange. It should be noted that as the index was modified since 6 December, 2008, price and cash return index refer to the previous period whereas total index is used from this date onwards. Since the data content employed in both of these indicators is similar (price changes and cash dividends) and the only difference lies in the base number, the two indicators have been aligned to solve the problem. In view of the above and the fact that the data related to price and cash return has been calculated since 25 July, 1999, based on the period under study in this article, the number of data used is 2988 daily observations.

The following table provides descriptive statistics of the logarithmic return of this index along with the corresponding histogram.

Table 1: Descriptive statistics and histogram of logarithmic return of the total index

Mean	0.001286
Median	0.000986
Maximum	0.06424
Minimum	-0.05450
Standard deviation	0.00556
Skewness	0.9167
Kurtosis	20.7789
Jarque-Bera Prob.	0.000



6.1 Executing Geometric Brownian Motion Model

Consider the Geometric Brownian stochastic model for asset price $S(t)$ with thrust sentence μ and volatility σ as follows:

$$dS = \mu S dt + \sigma S dz$$

where dz is the standard Brownian process (or Wiener). This equation can be rewritten as follows:

$$dlnS = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dz$$

If $\nu = \mu - \frac{\sigma^2}{2}$, then

$$S\left(t\right) = S(0)e^{(\nu t + \sigma \int\limits_0^t dz)}$$

To perform the asset price simulation over a period of $(0, t)$, time should be discretized in time steps of δt. To this end, using the last equation as well as the properties of the standard Wiener process, the following is obtained:

$$S\,(t+\delta t) = S(t)\exp(\nu\delta t + \sigma\sqrt{\delta t}\varepsilon)$$

where $\varepsilon \approx N(0,1)$ is the standard normal random variable.

According to the above equation, it is possible to generate sample paths for asset price using MATLAB software. This plan is introduced in the following profile to produce asset price sample paths:

```
Gray function SPaths=AssetPaths(S0,mu,sigma,T,NSteps,NRepl)
Gray SPaths=zeros(NRepl,NSteps);
Gray SPaths(:,1)=S0;
Gray dt=T/NSteps;

Gray sidt=sigma*sqrt(dt);
Gray for i=1:NRepl
Gray     for j=1:NSteps
Gray        Gray SPaths(i,j+1)=SPaths(i,j)*exp(nudt+sidt*randn);
Gray     end
Gray end
Gray end
Gray
```

The above plan, with the initial price (S_0), thrust (μ) , volatility (σ), execution time (T), time step (NSteps) and number of repetitions (NSteps), considers the displacement parameter (μ) as input, and then calculates v parameter and simulates the sample paths of Geometric Brownian motion. Thus, to perform Monte Carlo Simulation for a financial random variable, the only computational parameters are thrust (μ) and volatility (σ). To estimate the two parameters using Maximum Likelihood Estimation (MLE), the following equations will be obtained:

$$\hat{\mu} = \sum_{i=1}^{n} x_i/n, \hat{\sigma}^2 = \sum_{i}^{n} (x_i - \hat{\mu})^2/n$$

where $x_{t_i} := \log S_{t_i} - \log S_{t_{i-1}}$ is the logarithmic return efficiency of the variable under study, and is the number of data. Based on this, the estimated values of thrust and volatility parameters for the total index of Tehran Stock Exchange are 0.001286 and 0.005599, respectively. Now, using these estimated parameters, the total index can be simulated for the desired number of periods as well as for the number of sample paths. For example, 5 annual paths from the sample paths for this index with an initial value of 25905.6 (index value at the end of 2011) and a one-day time are simulated and displayed in the following figure. To do this, the following instructions should be simply followed:

```
Gray ■ randn('seed',0);
Gray ■ paths=AssetPaths(25905.6,0.001286,0.005599,1,365,5);
Gray ■ plot(paths')
Gray
```

By following the above instructions, the sample paths are drawn as follows:

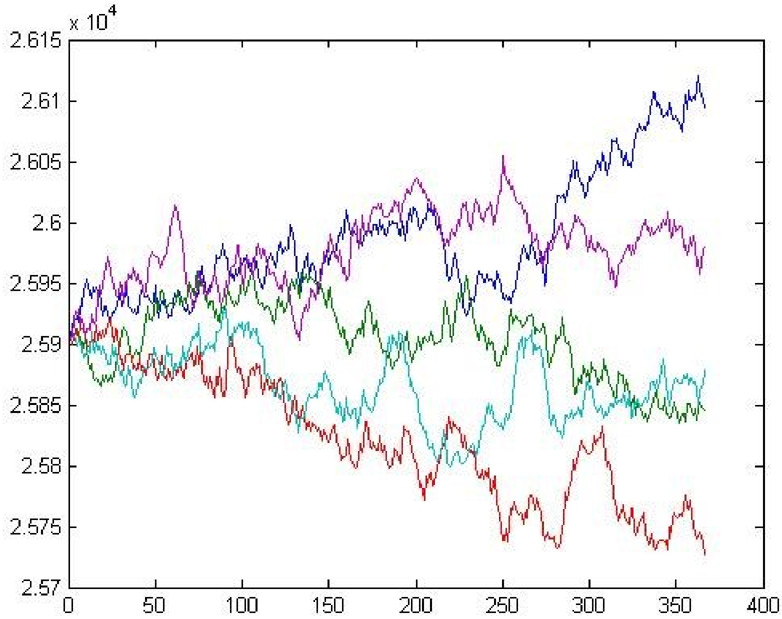


Figure 1: Simulation result of five sample paths of Geometric Brownian Motion for the total index

It should, however, be noted that although in the above figure only five paths were simulated for better representation, to use simulation techniques in practice, a large number of sample paths are created, and decisions are made based on the final results. For example, the following figure depicts the distribution function of the final simulation results of 10,000 one-year sample paths which can be easily used for future calculations such as pricing of derivatives, forecasting the desired variable, and risk management (for example calculating Value at Risk (VaR), etc.).

6.2 Executing Jump-Diffusion Model

One of the most widely used jump-diffusion models is the Merton model introduced in 1976. The model is a combination of Geometric Brownian model and a jump

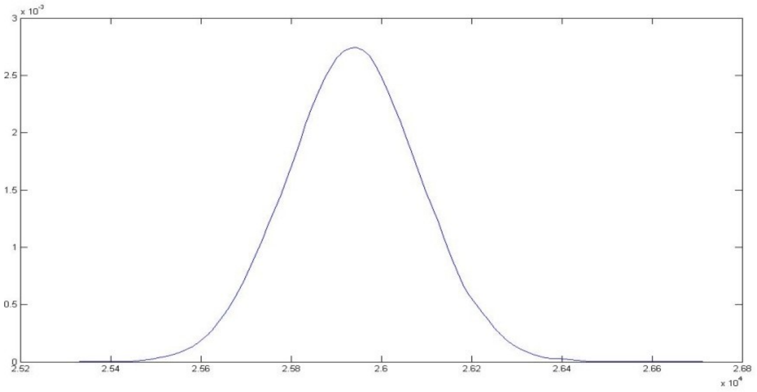


Figure 2: Distribution of the simulation of 10,000 sample paths of Geometric Brownian motion for the total index

sentence as follows:

$$dS_t = \mu S_t dt + \sigma S_t dW_t + S_t dJ_t$$

where J_t is a univariate jump process as follows

$$dJ_t = (Y_{N_t} - 1)dN_t$$

where $(N_t)_{T \geq 0}$ has Poisson distribution with density of λ. Also, Y_j indicates the size of the $-j^{th}$ jump. Mertons model is based on the assumption that Y_j has a distribution of $i.i.d$ and a logarithm of -normal. In other words:

$$Y_j \sim \exp(N(\mu_Y, \sigma_Y^2))$$

which is also independent of W. Maximum Likelihood Estimation (MLE) can also be used to estimate the parameters of the model. Results of estimating the parameters of the above model for the Tehran stock exchange total index are as follows:

Table 2: Descriptive statistics and histogram of logarithmic return of the total index

σ_Y	μ_Y	λ	σ	μ
0.00829	0.001346	0.277616	0.003049	0.001285

Based on these figures, the total index simulation operation can be performed for later use. The following figure demonstrates the distribution diagram obtained

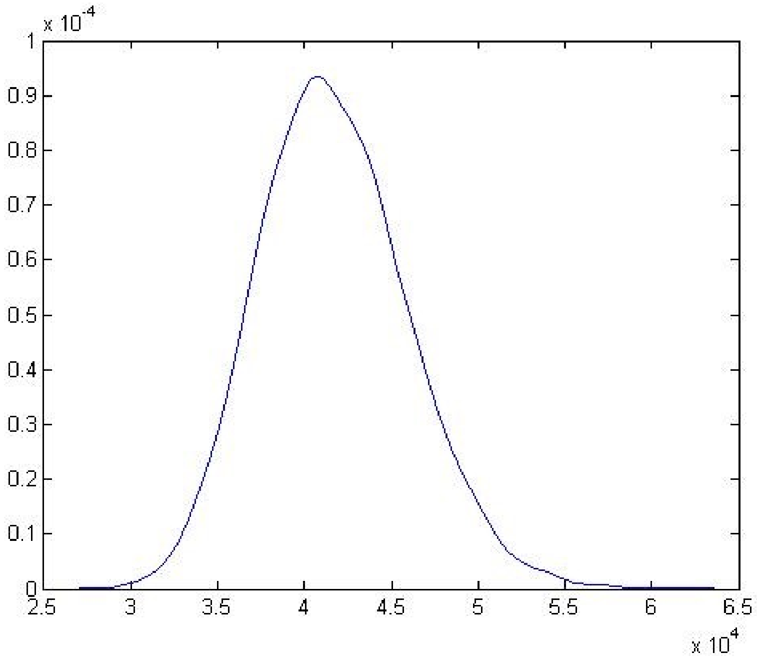


Figure 3: Distribution of simulation of 10,000 jump-diffusion sample paths for the total index

from the simulation of 10,000 sample paths of the model based on the estimated parameters of the total index.

As can be seen, the fat tail phenomenon is well observed in the distribution diagram thanks to the presence of jump sentence in the underlying jump diffusion process use to simulate the paths. Again, results of this distribution can be applied for further calculations, such as derivative pricing and risk measurement methods.

7 Option Pricing

In 1973, Black and Scholes and Merton revolutionized applied mathematical finance by introducing option pricing theory. Inspired by the idea of portfolio risk insurance using option bond, they provided a model for pricing European call option. To better understand this, consider the issuer of a European call option. If the underlying asset price is higher than the agreed price, the issuer will be exposed to risk, and as a result, he can purchase a certain amount of the underlying asset to cover the risk. On this basis, Black and Scholes demonstrated that through a special combination of underlying asset and call option in a portfolio, a risk-free portfolio can be created which in an efficient market based on the principle of no-arbitrage opportunity, will have returns equal to risk-free returns (interest rate).

By explicating the Black, Scholes and Merton model in this part, this article aims at suggesting a numerical method to solve the model, and implement it using MATLAB software.

7.1 Black, Scholes and Mertons Mathematical Model

As mentioned above, this section describes the Black, Scholes and Mertons model from mathematical perspective. To this end, let us suppose that function C is the price of a call option or a derivative, and S_t comprises its underlying asset. Therefore, should be a function based on the price of each stock in time t, i.e. $C = C(S_t, t)$. Also, let us assume that the underlying asset price S_t follows the stochastic Brownian motion process below:

$$dS_t = \mu S_t dt + \sigma S_t dz_t$$

where σ is the price volatility of each stock, μ is the return rate of the item, t is time, is time change, and is the standard Wiener or Brownie process. In addition, suppose that both parameters of and are constant. Now, in a portfolio with the sale of a unit of European stock call option and the purchase of underlying asset units of Δ_t, the value of this portfolio is denoted by $\Pi\left(S_t, t\right)$ symbol which is calculated in time t as follows:

$$\Pi = -C + \Delta_t S_t$$

where $C = C(S_t, t)$ is the call option value. Note that Δ_t changes at time t. In this case, the value change of this portfolio is as follows:

$$d\Pi\left(S_t, t\right) = -dC + \Delta_t dS_t$$

Without going into details and eliminating the time index to facilitate the process, it can be proved that the European call option price applies to the following model (Wilmott, 2006)

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = \circ\,, \quad \circ < S < \infty\,,\ \circ < t < T,$$

and

$$\begin{aligned}
&C(S,T) = \max\left(S - X, \circ\right) = \left(S - X\right)_+\,,\ \circ < S < \infty, \\
&C(\circ, t) = \circ\,, \quad \circ < t < T, \\
&C(S,t) \sim S - Xe^{-r(T-t)}\,, \quad \circ < t < T\,.
\end{aligned}$$

The above equation is a second-order linear partial differential equation known as the Black and Scholes partial differential equation. The equation, along with the initial and boundary conditions, is an initial and boundary value problem which is solved in the next section using finite-difference numerical method. Also, the proposed method is used for hypothetical data, and the results are presented.

7.2 Numerical Solution of Black and Scholes Pricing Model

While Black and Scholes equation has an analytical answer, through small changes in the equation such as changing the behavior model of underlying asset from Geometric Brownian motion model to a more complex one, the resulting equation will hardly have any analytical answer. Therefore, using numerical methods in such cases is necessary. Accordingly, in this section the Black and Scholes equation is solved using these methods.

To this end, first the scope of problem definition is written as follows:

$$D = \{(S, t)\,|\circ < S < S_{\max}, \ \ \circ < t < T\}$$

where S_{max} is a supremum for S (the highest observed stock price in the market) and T is the maturity time. Now, the $[\circ, S_{\max}]$ interval is divided by N subintervals, each with the length of $h = \frac{S_{\max}}{N}$, and k which denotes time step length. The domain above is then reticulated as follows:

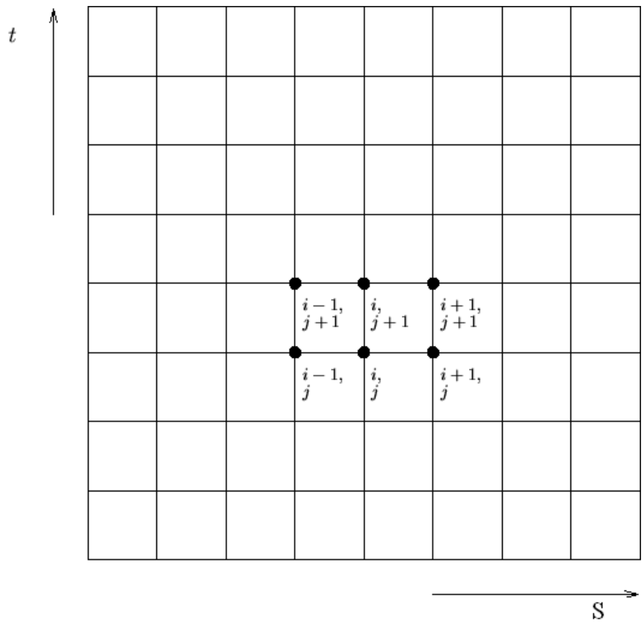


Figure 4: Price-time reticulation method

Now, using Taylor series expansion, derivatives in the differential equation of Black and Scholes at point $(S_{\mathrm{i}}, t_{\mathrm{j}})$ is approximated as follows:

$$\frac{\partial C}{\partial t} = \frac{C\left(S_{\mathrm{i}}, t_{\mathrm{j+1}}\right) - C\left(S_{\mathrm{i}}, t_{\mathrm{j}}\right)}{k} - \frac{k}{2}\frac{\partial^2 C}{\partial t^2}\left(S_{\mathrm{i}}, \eta_{\mathrm{j}}\right)$$

Also, $\frac{\partial C}{\partial S}$ and $\frac{\partial^2 C}{\partial S^2}$ are calculated using the central formula as follows:

$$\frac{\partial C}{\partial S} = \frac{C\left(S_{\mathrm{i}+1}, t_j\right) - C\left(S_{\mathrm{i}-1}, t_j\right)}{2h} - \frac{h^2}{6}\frac{\partial^3 C}{\partial S^3}\left(\xi_{\mathrm{i}}, t_j\right)$$
$$\frac{\partial^2 C}{\partial S^2} = \frac{C\left(S_{\mathrm{i}+1}, t_j\right) - 2C\left(S_{\mathrm{i}}, C_j\right) + C\left(S_{\mathrm{i}-1}, t_j\right)}{h^2} - \frac{h^2}{12}\frac{\partial^4 C}{\partial x^4}\left(\varsigma_{\mathrm{i}}, t_j\right)$$

where ς_{i} , ξ_{i} and η_j are dependent variables. Finally, by rewriting the Black and Scholes equation at point $(S_{\mathrm{i}}, t_{\mathrm{j}})$, the following is obtained:

$$\frac{\partial C\left(S_i, t_j\right)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C\left(S_i, t_j\right)}{\partial S^2} + rS\frac{\partial C\left(S_i, t_j\right)}{\partial S} - rC\left(S_i, t_j\right) = \circ$$

and by placing the above differential equations for the derivatives and eliminating error sentences, the following differential order is achieved:

$$\widehat{C}_{i,j+1} = -(\lambda S_i^2 + \gamma S_i)\widehat{C}_{i-1,j} + (1\ + 2\lambda S_i^2 + rk)\widehat{C}_{i,j} - (\lambda S_i^2 + \gamma S)\widehat{C}_{i+1,j}$$

where $i = 1, 2, 3, ..., N$, $j = 1, 2, 3, ...$ and $\lambda = \frac{k}{2h^2}\sigma^2$ and $\gamma = \frac{rk}{2h}$. Moreover, $\widehat{C}_{i,j}$ is the approximation of $C\left(S_{\mathrm{i}}, t_{\mathrm{j}}\right)$ with the following error:

$$\tau_{ij} = -\frac{k}{2}\frac{\partial^2 C}{\partial t^2}\left(S_{\mathrm{i}}, \eta_{\mathrm{j}}\right) - \frac{1}{2}\sigma^2 S^2 \frac{h^2}{12}\frac{\partial^4 C}{\partial x^4}\left(\xi_{\mathrm{i}}, t_j\right) - rS\frac{h^2}{6}\frac{\partial^3 C}{\partial S^3}\left(\varsigma_{\mathrm{i}}, t_j\right)$$

In conclusion, hypothetical data are applied to implement the method provided above through MATLAB software. For example, consider a put option with the initial price of 50 and the agreed price of 60 units, an interest rate of 0.06%, volatility of 0.2 and maturity of 3. By dividing the axes of and into 100 and 250 parts using the method above, a price equal to 6.94 units is obtained for this option. The relationship between price of option bond and time and price of the underlying asset is displayed in the following figure:

8 Conclusion and Suggestions

This study solved certain financial quantities using advanced mathematical methods and modeling as well as employing numerical concepts and MATLAB software. It was also demonstrated that the models were inefficient in certain cases, and methods to solve such inefficiencies were introduced through an example. Parameters of some of these models were then estimated through data of the total index of Tehran Stock Exchange, the index was simulated based on it, and its one-year distribution was extracted. Similarly, the paper explained option pricing models, and provided a numerical method for solving the Black and Scholes model.

Finally, given the central importance of presenting analytical models such as Black and Scholes model and solving new models in special cases, researchers interested in this field are suggested to focus on such issues. For example, Campolieti

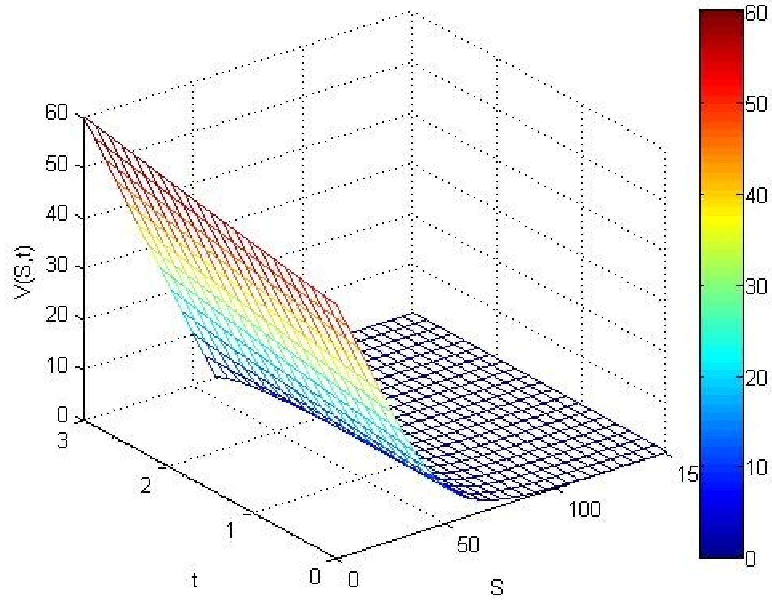


Figure 5: Relationship between option bond price and the time and price of the underlying asset

and his research team (Albaness and Campolieti, 2005; as well as Campolieti and Makarov, 2005 and 2006) studied a new family of integrable continuous diffusion models, and focused their attention on transition densities for several types of non-linear stochastic differential equations. Generally, the main reason for studying such alternative models is to completely and accurately retrieve the properties of asymmetry (skewness) or smiling in the markets implicit volatility. Researchers are also suggested to use different models such as Markov switching model and its combination with other models such as jump-diffusion model, and to localize and simulate these types of models according to the specific economic conditions found in Iran capital market.

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How to Cite: Moslem Peymany [1], *Mathematical modeling of stock price behavior and option valuation*, Journal of Mathematics and Modeling in Finance (JMMF), Vol. 1, No. 1, Pages:113–129, (2021).