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Research paper

Explicit solutions of Cauchy problems for degenerate hyperbolic equations with Transmutations methods

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Abstract:
This article's primary goal is to compute an explicit transmutation-based solution to a degenerate hyperbolic equation of second order in terms of time. To reduce a new problem to a problem that has already been solved, or at the very least to a smaller problem, is a standard mathematics strategy known as the transmutations method. similar to utilizing heat equations to solve wave equations. Using transmutation methods, we solve this problem using the well-known Kolmogorov equation. We present the solution of wave equations using transmutation methods and show that it is equivalent to the solution obtained by applying the Fourier transform in order to support our methodology.

Keywords: Degenerate Partial Differential Equations, Transmutation Methods, Kolmogorov Equation, Inverse Laplace Transform, Laplace Transform
MSC Classification: 42B37, 44A05, 44A10.

1 Introduction

The purpose of this article is to compute an explicit transmutation-based solution to the following degenerate hyperbolic partial differential equation (PDE):

$$\begin{cases} \partial_{tt}u - \Delta_x u - \langle x, \nabla_y u\rangle = 0 \ , & \mathbb{R}^n_x \times \mathbb{R}^n_y \times (0,\infty) \\ u(x,y,0) = 0 \ , & \partial_t u(x,y,0) = \psi(x,y) \end{cases} \tag{1}$$

This PDE is highly degenerate since it is missing the diffusive term $\Delta_y u$. The corresponding parabolic Cauchy problem was first introduced by Andrey Kolmogorov in a famous 1934 note [1]:

$$\begin{cases} \frac{\partial f}{\partial t} - \Delta_x f - \langle x, \nabla_y f\rangle = 0, & \mathbb{R}^{2n} \times (0,\infty) \\ f(x,y,0) = \psi(x,y), & (x,y) \in \mathbb{R}^{2n} \end{cases} \tag{2}$$

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Many believe Andrey Kolmogorov to be the greatest mathematician in Russian history, as well as one of the most brilliant mathematicians the world has ever seen. He was a guy of various interests, and thanks to his originality and cunning mind, he made contributions to a wide range of mathematical fields.

In his seminal 1934 paper [1] on Brownian motion Kolmogorov introduced the aforementioned highly degenerate equation 2 for which the Cauchy problem admits an explicit fundamental solution C^{∞} off the diagonal, thus effectively proving the hypoellipticity of his operator thirty years prior to Hormanders celebrated work. This equation is crucial to the kinetic theory of gases.

Let us take a brief detour into the context of "Kinetic Theory of Gases," which can be explained as follows. The kinetic theory of gases attempts to explain the microscopic properties of a gases, such as volume, pressure, and temperature, as well as transport properties such as viscosity, thermal conductivity and mass diffusivity in terms of the motion of its molecules. The gas is assumed to consist of a large number of identical, discrete particles called molecules, a molecule being the smallest unit having the same chemical properties as the substance.

The kinetic theory of gases was historically the first formal application of statistical mechanics concepts. Maxwell, Boltzmann, and Clausius developed key components of the kinetic theory between 1860 and 1880. Kinetic theories are available for gas, solid as well as liquid. [19]

Although the Cauchy problem (1) doesn't appear to have a well-developed theory, this PDE also does appear to have a direct application in the kinetic theory of gases for the study of the density of a system of N gas particles in the phase space that models the collision of particles in a specific surrounding bath, where the aggregation of particles induces friction contribution [18]. The density of particles with constant velocity one and location $(x, y) \in \mathbb{R}^{2n}$ at time t is represented by $u = u(x, y, t)$ in this equation, which is a member of a class of evolution equations emerging in the kinetic theory of gases. [17]

As mentioned, it appears that no explanation has yet been put forth to address the Cauchy problem (1). Therefore, we refer to the publications of two scientists named Berg and Detmand in 1968. They sought a method for deriving the answer to a partial derivative problem of the second order from the first order, and vice versa, in the article on problems relating to partial derivatives [2]. By employing the Laplace transform and Laplace inverse in this research, they were able to accomplish their objective under suitable boundary conditions. This is an important step in the transformation methods community.

Following that, they presented applications of their method in a subsequent article [3]. They solved the wave equations, radial wave equations, and eigenvalue solutions using a previously proven method. In another paper [4], they extended their research to find an arithmetic operator that calculates the solution of partial differential equations. In 1969, they broadened their research and developed their method for solving Drickel problems with initial values. In an article about a class of

Drickel problems and initial values [6], they introduced the transformation method and showed the validity of their method with an example.

Then in 1974, Ruben Hersh wrote a comprehensive article [5] about transformation methods. In this article, he mentioned different types of conversion methods and provided various examples. And he introduced the transformation method as a standard mathematical strategy to reduce it to a previously solved problem, or at least to a simpler one, when faced with a new problem. For example, it is possible to reduce a problem with unit coefficient to one with ordinary coefficients; reducing the problem containing a small parameter to a parameter independent of the parameter; Converting the second-order equation to the first-order equation or vice versa; And to convert a Gorsa problem to a Cauchy problem or vice versa, he pointed out. To say briefly, these authors demonstrated a number of relationships between solutions to first order and second order PDEs with initial boundary conditions in the articles [2], [3], and [4].

In the sight of these researches, we would like to present the theory on the explicit solution of 1, for which no theory has yet been developed. Furthermore, we will demonstrate that, under the suitable boundary conditions, the solution of the heat equation associated with the Laplace inverse conversion yields the solution of the initial value problem for the wave equation. To better understand the transmutation methods strategy, consider the following proposition, which finds the solution of the heat equation from the solution of the wave equation.

Proposition 1.1. ***From solution of Wave equation to solution of Heat equation***
Suppose we have a solution to the Cauchy problem of the wave equation:

$$\begin{cases} \partial_{tt}u - \Delta_x u = 0 \ , & \mathbb{R}^n \times (0, \infty) \\ u(x,0) = \varphi(x) \ , & \partial_t u(x,0) = 0 \end{cases} \tag{3}$$

And let $G(\sigma, t)$ as defined in the following be the heat kernel in the space variable $\sigma \in \mathbb{R}$ and time variable $t \in \mathbb{R}^{\geq 0}$:

$$G(\sigma, t) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{\sigma^2}{4t}} \tag{4}$$

Let us now define a function using the formula:

$$v(x,t) = \int_{\mathbb{R}} G(\sigma, t) u(x, \sigma) d\sigma = P_t(u(x,.))(0) \tag{5}$$

We claim that the function $v(x,t)$ defined by the solution of the wave equation is the solution of the heat equation:

$$\begin{cases} \partial_t v - \Delta_x v = 0 \ , & \mathbb{R}^n \times (0, \infty) \\ v(x,0) = \varphi(x) \end{cases} \tag{6}$$

Proof. Let us first demonstrate that heat kernel (4) is a solution of the following heat equation:

$$\begin{cases} \partial_t G - \Delta_\sigma G = 0 \ , \quad \mathbb{R} \times (0, \infty) \\ G(\sigma, 0) = 0 \ , \end{cases} \tag{7}$$

The first condition is self-evident. Let's compute its derivatives to prove the derivatives part:

$$\frac{\partial G}{\partial \sigma}(\sigma, t) = \frac{-2\sigma}{4t}(4\pi t)^{-\frac{1}{2}} e^{-\frac{\sigma^2}{4t}}$$
$$\frac{\partial^2 G}{\partial \sigma^2}(\sigma, t) = \frac{-2}{4t}(4\pi t)^{-\frac{1}{2}} e^{-\frac{\sigma^2}{4t}} + \frac{\sigma^2}{4t^2}(4\pi t)^{-\frac{1}{2}} e^{-\frac{\sigma^2}{4t}}$$
$$\frac{\partial G}{\partial t}(\sigma, t) = -\frac{1}{2} t^{-\frac{3}{2}}(4\pi)^{-\frac{1}{2}} e^{-\frac{\sigma^2}{4t}} + \frac{\sigma^2}{4t^2}(4\pi t)^{-\frac{1}{2}} e^{-\frac{\sigma^2}{4t}}$$

We can conclude from the above calculation that (4) satisfies (7). The derivative of $v(x, t)$ can now be computed using Fobini Theorem, integration by parts, and (7) as follows:

$$\frac{\partial v}{\partial t}(x, t) = \frac{\partial}{\partial t} \int_{\mathbb{R}} G(\sigma, t) u(x, \sigma) d\sigma = \int_{\mathbb{R}} \frac{\partial G}{\partial t}(\sigma, t) u(x, \sigma) d\sigma =$$
$$\int_{\mathbb{R}} \frac{\partial^2 G}{\partial \sigma^2}(\sigma, t) u(x, \sigma) d\sigma = \int_{\mathbb{R}} G(\sigma, t) \frac{\partial^2 u}{\partial \sigma^2}(x, \sigma) d\sigma$$

Due to the (3), we have $\frac{\partial^2 u}{\partial \sigma^2}(x, \sigma) = \Delta_x u$, therefore:

$$\frac{\partial v}{\partial t}(x, t) = \int_{\mathbb{R}} G(\sigma, t) \Delta_x u(x, \sigma) d\sigma =$$
$$\Delta_x \int_{\mathbb{R}} G(\sigma, t) u(x, \sigma) d\sigma = \Delta_x v(x, t)$$

This computation demonstrates that $v_t - \Delta_x v = 0$ in $\mathbb{R}^n \times (0, \infty)$. Furthermore, for the initial condition:

$$\lim_{t \to o^+} v(x, t) = \lim_{t \to o^+} \int_{\mathbb{R}} G(\sigma, t) u(x, \sigma) d\sigma =$$
$$\lim_{t \to o^+} P_t(u(x, .))(0) = u(x, 0) = \varphi(x)$$

This satisfies the Cauchy problem for the heat equation. □

In this paper, we want to proceed in the opposite direction, knowing that $v(x, t)$ is the solution to the Cauchy problem (75), and then find a new function $u(x, t)$

that solves the Cauchy problem (3). As previously stated, this is the transmutation technique, which employs solutions of other problems to find a solution to the current one.

For precision,let

$$D = (D_1, D_2, \cdots, D_n), x = (x_1, x_2, \cdots, x_n)$$

with

$$D_i = \frac{\partial}{\partial_{x_i}}$$

and let $P(x, D)$ be a finite order linear partial differential operator (usually elliptic type). Then consider the following pair of problems:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = P(x,D)u(x,t), & t > 0 \\ u(x,0) = \phi(x) & \\ B(x,D)u(x,t) = f(x,t) & x \in S, t > 0 \end{cases} \tag{8}$$

And

$$\begin{cases} \frac{\partial^2}{\partial t^2}v(x,t) = P(x,t)v(x,t) & t > 0 \\ v(x,0) = 0 \qquad v_t(x,0) = \phi(x) & \\ B(x,D)v(x,t) = g(x,t) & x \in S, t > 0 \end{cases} \tag{9}$$

The purpose of this paper will be to use the inverse Laplace transform to relate the solvability of problem (8) to the solvability of problem (9). The use of the Laplace transform imposes constraints on the functions $f(x,t)$ and $g(x,t)$, but these conditions are met in a wide range of applications. To preserve and demonstrate the technique's essential simplicity, we use strictly formal methods of proof. Most of the derived results are easily verified to hold in general. The basic theorem will be treated rigorously later.

In the following section, we will first use Fourier transforms and characteristic methods to solve the wave and Kolomogorov partial differential equations. In the following sections, we will prove the relationships between solutions of (8) and (9). Finally, an explanation of how to use the transmutation technique to solve the (1) is provided, along with evidence that the solution of the wave equation using the transmutation technique is equivalent to the solution using the Fourier transform and characteristics methods.

2 Classical Methods

In this chapter, we will solve the wave and Kolomogorov partial differential equations using the Fourier transform and characteristic methods. Let's proceed with the wave equation based on the book of Nicola Garofalo in [7].

2.1 Wave equation

Wave equation in n-dimentional space on a function f that lives in space-time $\mathbb{R}^{n+1}$ for given data φ and ψ is defined as follows:

$$\begin{cases} \partial_{tt} f - c^2 \Delta_x f = 0 \ , & \mathbb{R}^n \times (0, \infty) \\ f(x,0) = \varphi(x) \ , & \partial_t f(x,0) = \psi(x) \end{cases} \tag{10}$$

It is sufficient for our purposes to limit ourselves to solving this PDE for $n = 3$.

Theorem 2.1. *Let $\varphi \in C^3(\mathbb{R}^3), \psi \in C^2(\mathbb{R}^3)$, and define*

$$f(x,t) = \frac{1}{4\pi c^2 t^2} \int_{S(x,ct)} \{\varphi(y) + \langle \nabla\varphi(y), y - x\rangle + t\psi(y)\} d\sigma(y) \tag{11}$$

Then, $f \in C^2(\mathbb{R}^3 \times (0,\infty)) \cap C^1(\mathbb{R}^3 \times (0,\infty))$ and such function provides the unique solution to the Cauchy problem (10) for $n = 3$.

Fourier Transform: Let's take a brief detour into the meaning of the Fourier transform and its basic properties before moving on to the proof of this theorem. The Fourier transform of a function $g(x)$ is denoted by $\widehat{g}(\xi)$. With $\xi \in \mathbb{R}^3$ we denote the Fourier transform of function $g(x)$ as:

$$\widehat{g}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \langle \xi, x\rangle} g(x) dx \tag{12}$$

Fourier transform is a linear operator with the following properties:

- If translation operator is defined as $T_h(x) = x + h, h \in \mathbb{R}^n$ then :

$$\widehat{T_h g}(\xi) = e^{2\pi i \langle \xi, h\rangle} \widehat{g}(\xi) \tag{13}$$

- If dilation operator is defined as $\delta_\lambda(x) = \lambda x, \lambda > 0$ then :

$$\widehat{\delta_\lambda g}(\xi) = \lambda^{-n} \widehat{g}(\frac{\xi}{\lambda}) \tag{14}$$

- Fourier transform derivatives can be simplified as follows:

$$\widehat{\partial_{x_i} g}(\xi) = 2\pi i \xi_i \widehat{g}(\xi) \tag{15}$$

$$\widehat{\partial_{x_i x_j} g}(\xi) = -4\pi^2 \xi_i \xi_j \widehat{g}(\xi) \tag{16}$$

- Another important property of Fourier transform is:

$$\widehat{x_i g}(\xi) = -\frac{1}{2\pi i} \partial_{\xi_i} \widehat{g}(\xi) \tag{17}$$

Proof. To prove theorem (2.1), we must first apply the Fourier transform to the PDE (10). Therefore, with $\xi \in \mathbb{R}^3$ we denote the Fourier transform of function $f(x,t)$ as:

$$\widehat{f}(\xi,t) = \int_{\mathbb{R}^3} e^{-2\pi i\langle \xi,x\rangle} f(x,t)dx \tag{18}$$

Now we apply the Fourier transform to the PDE (10) for $n = 3$:

$$\begin{cases} \widehat{\partial_{tt} f} - \widehat{c^2\Delta_x f} = 0 \ , & \mathbb{R}^3 \times (0,\infty) \\ \widehat{f}(\xi,0) = \widehat{\varphi}(\xi) \ , & \widehat{\partial_t f}(\xi,0) = \widehat{\psi}(\xi) \end{cases} \tag{19}$$

Based on the properties of Fourier transform, we can compute:

- $\widehat{\partial_{tt} f} = \partial_{tt}\widehat{f}$. Because Fourier is with respect to the space variable, we can use Fobini's theorem to extract the derivative with respect to time from the Fourier transform.
- $\widehat{c^2\Delta_x f} = c^2 \sum_{i=1}^3 \widehat{\partial_{x_i x_i} f} = c^2 \sum_{i=1}^3 (2\pi i\xi)^2 \widehat{f} = -4\pi^2 c^2 |\xi|^2 \widehat{f}$

Therefore, we can rewrite (19) like:

$$\begin{cases} \partial_{tt}\widehat{f} + 4\pi^2 c^2 |\xi|^2 \widehat{f} = 0 \ , \\ \widehat{f}(\xi,0) = \widehat{\varphi}(\xi) \ , \qquad \widehat{\partial_t f}(\xi,0) = \widehat{\psi}(\xi) \end{cases}$$

If we fix variable $\xi \in \mathbb{R}^3$, then it's possible to change variable $\widehat{f}(\xi,t) = y(t)$ and obtain the following ordinary differential equation:

$$\begin{cases} y''(t) + 4\pi^2 c^2 |\xi|^2 y(t) = 0 \ , \\ y(0) = \widehat{\varphi}(\xi) \ , \qquad y'(0) = \widehat{\psi}(\xi) \end{cases}$$

This is a harmonic oscillator case of an ODE with the following solution:

$$y(t) = A\cos(2\pi c|\xi|t) + B\sin(2\pi c|\xi|t)$$

We can compute A and B based on the initial conditions and obtain:

$$y(t) = \widehat{\varphi}(\xi)\cos(2\pi c|\xi|t) + \frac{\widehat{\psi}(\xi)}{2\pi c|\xi|}\sin(2\pi c|\xi|t) \tag{20}$$

Fourier transform of the measure carried by a sphere: Before continuing the proof of the theorem, let's take another detour. According to the book [7], the Fourier transform of the measure carried by a sphere is defined as follows:

$$\widehat{d\sigma_R} = \int_{S_R} e^{-2\pi i\langle \xi,x\rangle} d\sigma(x), \qquad S_R = \{x \in \mathbb{R}^3 s.t|x| = R\} \tag{21}$$

This is a spherically symmetric function whose value can be computed and is as follows:

$$\widehat{d\sigma_R} = 4\pi R^2 \frac{\sin(2\pi|\xi|R)}{2\pi|\xi|R}, \qquad \forall \xi \in \mathbb{R}^3 \tag{22}$$

To simplify the process, let's use the following notation:

$$\widehat{D\sigma_R} = \widehat{\frac{1}{4\pi^2 R^2} d\sigma_R}(\xi) = \frac{\sin(2\pi|\xi|R)}{2\pi|\xi|R} \tag{23}$$

Returning to the theorem proof, in (20) we can replace $\frac{\sin(2\pi|\xi|ct)}{2\pi|\xi|c}$ by $t\widehat{D\sigma_{ct}} = \widehat{tD\sigma_{ct}}$ and

$$\cos(2\pi|\xi|ct) = \frac{d}{dt}\frac{\sin(2\pi|\xi|ct)}{2\pi|\xi|c} = \frac{d}{dt}\widehat{tD\sigma_{ct}}(\xi)$$

As a result, we can rewrite the right hand side of the (20) as follows:

$$y(t) = \widehat{\varphi}(\xi)\frac{d}{dt}\widehat{(tD\sigma_{ct})}(\xi) + \widehat{\psi}(\xi)\widehat{tD\sigma_{ct}}(\xi) \tag{24}$$

Convolution and spherical average: Let's take a last detour before concluding the theorem's proof. Convolution of two function $g(x)$ and $v(x)$ is:

$$g(x) * v(x) = \int_{\mathbb{R}^n} g(t)v(x-t)dt \tag{25}$$

There is a crucial theorem about convolution that proves

$$g(x) * v(x) = v(x) * g(x) \tag{26}$$

Furthermore, a critical property of the Fourier transform regarding convolution states that if $g, v \in L^1(\mathbb{R}^n)$, then

$$\widehat{g * v} = \widehat{g}\widehat{v} \tag{27}$$

In addition, there is a key principle known as the metaprinciple, which states:

$$\widehat{g} = \widehat{v} \Leftrightarrow g = v \tag{28}$$

Having said that, let us define the spherical average a function $g : \mathbb{R}^n \to \mathbb{R}$ denoted by $M_g(x,r)$ as follows:

$$M_g(x,r) = \frac{1}{\sigma_{n-1} r^{n-1}} \int_{S(x,r)} g(y)d\sigma(y) \tag{29}$$

such that the sphere $S(x,r)$ has the following definition:

$$S(x,r) = \{y \in \mathbb{R}^n \ s.t \ |x-y| \le r\}$$

and σ_{n-1} is the area of sphere equal to

$$\sigma_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{30}$$

which $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is the Euler Gamma function. The key characteristic of these definitions based on the book [7] is that:

$$(g * D\sigma_R)(x) = M_g(x, R) \tag{31}$$

Returning to the theorem proof based on the key feature (31), we can rewrite right hand side of equation (24) as follows:

$$y(t) = \frac{d}{dt}\big(\widehat{tM_\varphi(., ct)}\big)(\xi) + \big(\widehat{tM_\psi(., ct)}\big)(\xi)$$

Now based on the metaprinciple (28) and applying the derivatives, we have:

$$u(x,t) = M_\varphi(x, ct) + t\frac{d}{dt}\big(M_\varphi(x, ct)\big) + tM_\psi(x, ct) \tag{32}$$

Finally, we obtain the solution (11) by calculation the spherical averages and derivatives which completes the proof. □

2.2 Kolmogorov Equation

In this section, we demonstrate how, by combining the Fourier transform with the method of characteristics, we can provide an analytical solution to the Cauchy problem for the Kolmogorov equation from [7]. In what follows, we will use the ordered couple (x, y), such that $x, y \in \mathbb{R}^n$ to represent the generic point in $\mathbb{R}^{2n}$ in the following Cauchy problem:

$$\begin{cases} \partial_t f - \Delta_x f - \langle x, \nabla_y f\rangle = 0, & \mathbb{R}^{2n} \times (0, \infty) \\ f(x, y, 0) = \psi(x, y), & (x, y) \in \mathbb{R}^{2n} \end{cases} \tag{33}$$

with $\psi \in C_0(\mathbb{R}^{2n})$. Due to the missing diffusion term $\Delta_y f$, equation (33) is highly degenerate. Let's find the solution of this hypoelliptic PDE. Suppose $\xi, \eta \in \mathbb{R}^n$, and we denote the dual variable of the point $(x, y) \in \mathbb{R}^{2n}$ in the partial Fourier transform by $(\xi, \eta) \in \mathbb{R}^{2n}$:

$$\widehat{f}(\xi, \eta, t) = \int_{\mathbb{R}^{2n}} e^{-2\pi i(\langle \xi, x\rangle + \langle \eta, y\rangle)} f(x, y, t)dxdy \tag{34}$$

We now proceed similarly to what we did for wave equation in previous section and apply to Kolmogrov PDE (33) a partial Fourier transform with respect to (x, y):

$$\begin{cases} \widehat{\partial_t f} - \widehat{\Delta_x f} - \widehat{\langle x, \nabla_y f\rangle} = 0, & \mathbb{R}^{2n} \times (0, \infty) \\ \widehat{f}(\xi, \eta, 0) = \widehat{\psi}(\xi, \eta), & (\xi, \eta) \in \mathbb{R}^{2n} \end{cases} \tag{35}$$

Each term in this system can be simplified using the Fourier properties (15), (16), and (17). In the previous section, we demonstrated how to simplify $\widehat{\partial_t f}$ and $\widehat{\Delta_x f}$. Now, let's compute $\widehat{\langle x, \nabla_y f\rangle}$:

$$\begin{aligned}\widehat{\langle x, \nabla_y f\rangle} &= \sum_{i=1}^{n} \widehat{x_i \partial_{y_i} f} = -\frac{1}{2\pi i}\sum_{i=1}^{n} \partial_{\xi_i}(\widehat{\partial_{y_i} f}) = -\sum_{i=1}^{n} \partial_{\xi_i}(\eta_i \widehat{f}) \\ &= -\sum_{i=1}^{n} \eta_i \partial_{\xi_i} \widehat{f} = -\langle \eta, \nabla_\xi \widehat{f}\rangle\end{aligned}$$

Now we can rewrite (35) as follows:

$$\begin{cases}\partial_t \widehat{f} + 4\pi^2|\xi|^2 \widehat{f} + \langle \eta, \nabla_\xi \widehat{f}\rangle = 0, \\ \widehat{f}(\xi, \eta, 0) = \widehat{\psi}(\xi, \eta)\end{cases}$$

By change of variable $\widehat{f}(\xi, \eta, t) = v(\xi, \eta, t)$, we obtain the following corresponding Cauchy problem:

$$\begin{cases}\partial_t v + 4\pi^2|\xi|^2 v + \langle \eta, \nabla_\xi v\rangle = 0, & \mathbb{R}^{2n} \times (0, \infty) \\ v(\xi, \eta, 0) = \widehat{\psi}(\xi, \eta), & (\xi, \eta) \in \mathbb{R}^{2n}\end{cases} \tag{36}$$

We employ characteristic methods to solve this PDE. To do so, consider the vector field $\tilde{V}(\xi, \eta, t) = (\eta, 0, 1)$ in $\mathbb{R}^{2n+1}$, which yields:

$$\left\langle \nabla_{(\xi,\eta,t)} v, \tilde{V} \right\rangle = -4\pi^2|\xi|^2 v \tag{37}$$

Starting at the point $(\xi, \eta, t) \in \mathbb{R}^{2n+1}$, we now construct the characteristic lines meaning that the following curves

$$\begin{gathered}\alpha : \mathbb{R} \to \mathbb{R}^{2n+1} \\ \alpha(s) = (\xi(s), \eta(s), t(s))\end{gathered}$$

Are solutions of the Cauchy problem for the following vector-valued ODE:

$$\alpha'(s) = \tilde{V}(\alpha(s)), \qquad \alpha(0) = (\xi, \eta, t)$$

This system can be written as:

$$\begin{cases}\xi'(s) = \eta(s), \\ \xi(0) = \xi\end{cases} \qquad \begin{cases}\eta'(s) = 0, \\ \eta(0) = \eta\end{cases} \qquad \begin{cases}t'(s) = 1, \\ t(0) = t\end{cases}$$

With the following solution:

$$\xi(s) = \xi + s\eta, \quad \eta(s) = \eta, \quad t(s) = t + s$$

We can easily see that $\theta(s)$ is satisfied by the following linear ODE if we define $\theta(s) = v(\alpha(s))$ and use the (37):

$$\theta'(s) = \left\langle \nabla_{(\xi,\eta,t)} v(\alpha(s)), \tilde{V}(\alpha(s)) \right\rangle = -4\pi^2 |\xi(s)|^2 \theta(s) = -4\pi^2 |\xi + s\eta|^2 \theta(s)$$

which shows that a constant A exists such that

$$\theta(s) = A e^{-4\pi^2 \int_0^s |\xi+\tau\eta|^2 d\tau}$$

To compute A, notice that

$$\theta(-t) = \widehat{f}(\xi - t\eta, \eta, 0) = \widehat{\psi}(\xi - t\eta, \eta)$$

Therefore,

$$A = \widehat{\psi}(\xi - t\eta, \eta) e^{-4\pi^2 \int_0^t |\xi-\tau\eta|^2 d\tau}$$

Now using the condition $\theta(0) = \widehat{f}(\xi, \eta, t)$, we obtain:

$$\widehat{f}(\xi, \eta, t) = \widehat{\psi}(\xi - t\eta, \eta) e^{-4\pi^2 \int_0^t |\xi-\tau\eta|^2 d\tau} \tag{38}$$

We want to recover f from such an explicit formula. We should be able to express the right-hand side of (38) as a Fourier transform using the following algebraic facts:

Lemma 2.2. *Let $A \in M_{n\times n}$, $B \in M_{n\times m}$, $C \in M_{m\times n}$, and $D \in M_{m\times m}$ and assume that A be invertible. Consider the partitioned matrix*

$$K = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

One has

$$\det(K) = \det(A). \det\left((D - CA^{-1} - B)\right)$$

If furthermore $A, B, C - DB^{-1}A$, and $D - CA^{-1}B$ are invertible, then

$$K^{-1} = \begin{pmatrix} A^{-1} - (C - DB^{-1}A)^{-1}CA^{-1} & (C - DB^{-1}A)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}$$

If instead $A, C, B - AC^{-1}D$, and $D - CA^{-1}B$, are invertible, then

$$K^{-1} = \begin{pmatrix} A^{-1} - A^{-1}B(B - AC^{-1}D)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ (B - AC^{-1}D)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}$$

Now define the linear operator $T(t) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ for $t \in \mathbb{R}$ as follows:

$$T(t) = \begin{pmatrix} I_n & O_n \\ -tI_n & I_n \end{pmatrix}$$

where the blocks I_n and O_n are the identity and zero matrices in $\mathbb{R}^n$. From Lemma (2.2) we have $det(T(t)) \equiv 1$ for every $t \in \mathbb{R}$ and

$$T(t)^{-1} = \begin{pmatrix} I_n & O_n \\ tI_n & I_n \end{pmatrix}$$

It is worth noting that the following notation A^t represents the matrix A's transpose. We have for the adjoint of $T(t)$,

$$T(t)^t(\xi, \eta) = (\xi - t\eta, \eta) \tag{39}$$

Now let's take a detour here and learn the following proposition from [7] chapter 5, which is required to compute the initial condition of (35).

Proposition 2.3. *Let us indicate with $Gl(n)$, the collection of all invertible linear mappings on $\mathbb{R}^n$. Also, We indicate with $O(n)$ all orthogonal transformations of $\mathbb{R}^n$ onto itself. Now assume that $A \in Gl(n)$. If $f \in L^1(\mathbb{R}^n)$, then one has:*

$$\widehat{f}(A^t\xi) = |det(A)|^{-1}\widehat{f \circ A^{-1}}(\xi) \tag{40}$$

In particular, if $T \in O(n)$, then

$$\widehat{f \circ T} = \widehat{f} \circ T$$

and therefore the Fourier transform of a spherically symmetric function is itself a spherically symmetric function.

Now by applying this proposition to (39), we obtain:

$$\widehat{\psi}(\xi - t\eta, \eta) = \widehat{\psi \circ T(t)^{-1}}(\xi, \eta) \tag{41}$$

Therefore, we can rewrite (38) as follows:

$$\widehat{f}(\xi, \eta, t) = \widehat{\psi \circ T(t)^{-1}}(\xi, \eta)e^{-4\pi^2 \int_0^t |\xi - \tau\eta|^2 d\tau} \tag{42}$$

Consider the following degenerate matrix to find a Fourier equivalent for the exponential term on the right hand side of (42):

$$Q = \begin{pmatrix} I_n & O_n \\ O_n & O_n \end{pmatrix}$$

Therefore,

$$QT(t)^t(\xi,\eta) = Q(\xi - t\eta, \eta) = (\xi - t\eta, 0)$$

Now, we can compute the $|\xi - \tau\eta|^2$ using these calculations.

$$|\xi - \tau\eta|^2 = \left\langle QT(\tau)^t(\xi,\eta), T(\tau)^t(\xi,\eta)\right\rangle = \left\langle T(\tau)QT(\tau)^t(\xi,\eta), (\xi,\eta)\right\rangle \tag{43}$$

Thus, we can rewrite the exponential term in (42) as follows:

$$\begin{aligned} e^{-4\pi^2\int_0^t |\xi-\tau\eta|^2 d\tau} &= e^{-4\pi^2\int_0^t \left\langle T(\tau)QT(\tau)^t(\xi,\eta),(\xi,\eta)\right\rangle d\tau} \\ &= e^{-4\pi^2\left\langle\left(\int_0^t T(\tau)QT(\tau)^t d\tau\right)(\xi,\eta),(\xi,\eta)\right\rangle} \end{aligned}$$

Now if we define the following matrix:

$$C(t) = \int_0^t T(\tau)QT(\tau)^t d\tau = \begin{pmatrix} tI_n & -\frac{t^2}{2}I_n \\ -\frac{t^2}{2}I_n & \frac{t^3}{3}I_n \end{pmatrix}$$

Even though matrix $T(\tau)QT(\tau)^t = \begin{pmatrix} I_n & -\tau I_n \\ -\tau I_n & \tau^2 I_n \end{pmatrix}$ is not invertible for any $\tau > 0$, from lemma (2.2) we get:

$$\det C(t) = \left(\frac{1}{3} - \frac{1}{4}\right)^n t^{4n} = 12^{-n}t^{4n} > 0 \qquad \forall t > 0 \tag{44}$$

Therefore, if we define

$$K(t) = t^{-1}C(t) = \begin{pmatrix} I_n & -\frac{t}{2}I_n \\ -\frac{t}{2}I_n & \frac{t^2}{3}I_n \end{pmatrix} \tag{45}$$

Now in order to compute the Fourier transform of the term

$$\frac{1}{\sqrt{\det K(t)}}(4\pi t)^{-\frac{2n}{2}} e^{-\frac{\left\langle K(t)^{-1}.,.\right\rangle}{4t}}$$

we need the following theorem from [7] chapter 5.

Theorem 2.4. *For any $t > 0$ and $\xi \in \mathbb{R}^n$ one has*

$$\left(e^{-4\pi^2 t|.|^2}\right)\hat{}(\xi) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|\xi|^2}{4t}} \tag{46}$$

On the other hand, one also has the following inverse formula

$$\left(\widehat{(4\pi t)^{-\frac{n}{2}} e^{-\frac{|.|^2}{4t}}}\right)_{x\to\xi}(\xi) = e^{-4\pi^2 t|\xi|^2} \tag{47}$$

More in general, if $A \in M_{n\times n}(\mathbb{R}^n)$ is a matrix such that $A^t = A$, and $A > 0$, then

$$\left(e^{-4\pi^2 t\langle A.,.\rangle}\right)\hat{}(\xi) = \frac{1}{\sqrt{\det A}}(4\pi t)^{-\frac{n}{2}} e^{-\frac{\left\langle A^{-1}\xi,\xi\right\rangle}{4t}} \tag{48}$$

and thus the inverse is

$$\left(\frac{1}{\sqrt{\det A}}\widehat{(4\pi t)^{-\frac{n}{2}}} e^{-\frac{\langle A^{-1}.,.\rangle}{4t}}\right)_{x\to\xi}(\xi) = e^{-4\pi^2 t\langle A\xi,\xi\rangle} \tag{49}$$

Returning to our computations and applying (49), we can conclude that

$$\begin{aligned}\left(\frac{1}{\sqrt{\det K(t)}}\widehat{(4\pi t)^{-\frac{2n}{2}}} e^{-\frac{\langle K(t)^{-1}.,.\rangle}{4t}}\right)(\xi,\eta) &= e^{-4\pi^2 t\langle K(t)(\xi,\eta),(\xi,\eta)\rangle} \\ &= e^{-4\pi^2\int_0^t|\xi-\tau\eta|^2 d\tau}\end{aligned} \tag{50}$$

With these calculations and convolution properties of Fourier transform, we can rewrite (42) as follows:

$$\begin{aligned}\widehat{f}(\xi,\eta,t) = \widehat{\psi\circ T(t)^{-1}}(\xi,\eta)\left(\frac{1}{\sqrt{\det K(t)}}\widehat{(4\pi t)^{-\frac{2n}{2}}} e^{-\frac{\langle K(t)^{-1}.,.\rangle}{4t}}\right)(\xi,\eta) = \\ \left((\psi\circ T(t)^{-1}) * \frac{1}{\sqrt{\det K(t)}}\widehat{(4\pi t)^{-\frac{2n}{2}}} e^{-\frac{\langle K(t)^{-1}.,.\rangle}{4t}}\right)(\xi,\eta)\end{aligned}$$

Now that we have this formula, we can apply the metaprinciple and obtain:

$$f(x,y,t) = \frac{(4\pi)^{-n}}{\sqrt{\det tK(t)}}\int_{\mathbb{R}^{2n}} e^{-\frac{\langle K(t)^{-1}(x-x',y-y'),(x-x',y-y')\rangle}{4t}}\psi\circ T(t)^{-1}(x',y')dx'dy' \tag{51}$$

With the following change of variable

$$(\bar{x},\bar{y}) = T(t)^{-1}(x',y')$$

such that

$$(x',y') = T(t)(\bar{x},\bar{y}) = (\bar{x},\bar{y}-t\bar{x})$$

and $dx'dy' = d\bar{x}d\bar{y}$, we finally obtain

$$f(x,y,t) = \frac{(4\pi)^{-n}}{\sqrt{\det tK(t)}}\int_{\mathbb{R}^{2n}} e^{-\frac{\left\langle K(t)^{-1}(x-\bar{x},y-\bar{y}+t\bar{x}),(x-\bar{x},y-\bar{y}+t\bar{x})\right\rangle}{4t}}\psi(\bar{x},\bar{y})d\bar{x}d\bar{y} \tag{52}$$

This formula (52) solves the Cauchy problem (33). To obtain a simplified version of this formula, first compute the inverse form of the matrix $K(t)$:

$$K(t)^{-1} = \begin{pmatrix} 4I_n & \frac{6}{t}I_n \\ \frac{6}{t}I_n & \frac{12}{t^2}I_n \end{pmatrix} \tag{53}$$

And therefore we easily can compute the exponential power as follows:

$$\begin{aligned}
&\left\langle K(t)^{-1}\left(x-\bar{x}, y-\bar{y}+t\bar{x}\right)\left(x-\bar{x}, y-\bar{y}+t\bar{x}\right)\right\rangle \\
=&4\left(|x-\bar{x}|^2+\frac{3}{t}\left\langle x-\bar{x}, y-\bar{y}+t\bar{x}\right\rangle+\frac{3}{t^2}\left|y-\bar{y}+t\bar{x}\right|^2\right) \\
=&4\left(\frac{|x-\bar{x}|^2}{4}+\frac{3}{t^2}\left|y-\bar{y}+t\bar{x}+\frac{t}{2}(x-\bar{x})\right|^2\right) \\
=&|x-\bar{x}|^2+12\left|\frac{y-\bar{y}}{t}+\frac{x+\bar{x}}{2}\right|^2
\end{aligned}$$

Also the coefficient term can be computed as follows:

$$\frac{(4\pi)^{-n}}{\sqrt{\det tK(t)}} = \frac{(4\pi)^{-n}}{\sqrt{12^{-n}t^{4n}}} = \frac{3^{\frac{n}{2}}}{(2\pi)^n}t^{-2n}$$

Now if we define function $p(x, y, \bar{x}, \bar{y}, t)$ like:

$$p(x, y, \bar{x}, \bar{y}, t) = \frac{3^{\frac{n}{2}}}{(2\pi)^n}t^{-2n}exp\left\{-\frac{1}{4t}\left(|x-\bar{x}|^2+12\left|\frac{y-\bar{y}}{t}+\frac{x+\bar{x}}{2}\right|^2\right)\right\} \tag{54}$$

Finally, the simplified version of the final solution is as follows:

$$f(x, y, t) = \int_{\mathbb{R}^{2n}} p(x, y, \bar{x}, \bar{y}, t)\psi(\bar{x}, \bar{y})d\bar{x}d\bar{y} \tag{55}$$

3 Transmutation Methods

As previously stated, the main goal of this article is to present an explicit solution for the degenerate hyperbolic equation (1) by employing transmutations methods that reduce the problem to a previously solved problem, or at least to a simpler problem, which in our case is the parabolic Kolmogorov Cauchy problem (2). To accomplish this, we use a particular transmutation technique that was implemented in [2], [3], and [6]. Suppose that

$$D = (D_1, D_2, \cdots, D_n) \qquad x = (x_1, x_2, \cdots, x_n) \qquad D_i = \frac{\partial}{\partial_{x_i}}$$

And define the following multi-index:

$$D^{\alpha} = D_1^{\alpha_1}, D_2^{\alpha_2}, \cdots D_n^{\alpha_n},$$

such that

$$P(x, D) = \sum_{\alpha: 0 \leq |\alpha| \leq m} a_{\alpha}(x) D^{\alpha}$$

Where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $a_{\alpha}(x)$ are given functions of x. Finally, let $S(x) = 0$ denote a cylindrical surface in (x, t) space and $B(x, D)$ a linear nontangential boundary operator whose domain is the manifold $S(x) = 0$. The smoottmess required of this cylinder will depend upon the operator $B(x, D)$. Now consider the following initial boundary value problems of the form:

$$\begin{cases} \partial_t u(x,t) = P(x,D)u(x,t), & t > 0 \\ u(x,0) = \phi(x) \\ B(x,D)u(x,t) = f(x,t), & x \in S, \ t > 0 \end{cases} \tag{56}$$

And

$$\begin{cases} \partial_{tt} v(x,t) = P(x,D)v(x,t), & t > 0 \\ v(x,0) = 0, \qquad v_t(x,0) = \phi(x) \\ B(x,D)v(x,t) = g(x,t), & x \in S, \ t > 0 \end{cases} \tag{57}$$

For the time being, we will assume that $\phi(x)$ has continuous derivatives of sufficient higher order to ensure the existence and continuity of $B(x, D)\phi(x)$ and $P(x, D)\phi(x)$. Finally, we assume that on $S(x) = 0$, $\phi(x)$ and all of its derivatives, up to those involved in $B(x, D)$, vanish.

The focus of this paper will be on relating the solvability of (56) to the solvability of (57) using the inverse Laplace transform based on the proofs provided in [2], [3] and [6]. The use of the Laplace transform will inevitably impose constraints on the functions $f(x, t)$ and $g(x, t)$, but these conditions are met in many applications. The inverse Laplace transform of the function $\psi(x, s)$ will be denoted by the following symbol:

$$L_s^{-1} \psi(x, s)_{s \to \tau}$$

Where s represents the variable in the transformed function $\psi(x, s)$, and τ represents the variable in the inverted function.

Theorem 3.1. *If $u(x, t)$ solves (56) and*

$$g(x,t) = \Gamma\left(\frac{3}{2}\right) L_s^{-1} \left\{ s^{-\frac{3}{2}} f\left(x, \frac{1}{4s}\right) \right\}_{s \to t^2} \tag{58}$$

Then the following $v(x, t)$ solves (57):

$$v(x,t) = \Gamma\left(\frac{3}{2}\right) L_s^{-1} \left\{ s^{-\frac{3}{2}} u\left(x, \frac{1}{4s}\right) \right\}_{s \to t^2} \tag{59}$$

provided the inverse Laplace transform exists.

The inverse of this theorem is provided in the following.

Theorem 3.2. *If $v(x,t)$ solves (57) and*

$$f(x,t) = \frac{1}{2\sqrt{\pi}t^{\frac{3}{2}}}\int_0^\infty \xi e^{-\frac{\xi^2}{4t}} g(x,\xi)d\xi \tag{60}$$

Then the following $u(x,t)$ solves (56):

$$u(x,t) = \frac{1}{2\sqrt{\pi}t^{\frac{3}{2}}}\int_0^\infty \xi e^{-\frac{\xi^2}{4t}} v(x,\xi)d\xi \tag{61}$$

provided the integrals exist for $t > 0$.

***Proof of theorem* (3.1) *and* (3.2).** Problems (56) and (57) will be reduced to the same problem using variable transformations and the introduction of the Laplace transform. To begin, we change variables $u(x,t)$ and $v(x,t)$ as follows:

$$\begin{aligned} u(x,t) =& u^*(x,t) + \phi(x) \\ v(x,t) =& v^*(x,t) + t\phi(x) \end{aligned}$$

Then (56) and (57) transform, respectively, into the problems

$$\begin{cases} u_t^*(x,t) = P(x,D)u^*(x,t) + P(x,D)\phi(x) & u^*(x,0) = 0 \\ B(x,D)u^*(x,t)|_s = f(x,t) & (\text{since } B(x,D)\phi(x) = 0 \text{ on } S) \end{cases} \tag{62}$$

And

$$\begin{cases} v_{tt}^*(x,t) = P(x,D)v^*(x,t) + tP(x,D)\phi(x) \\ v^*(x,0) = 0 \qquad v_t^*(x,0) = 0 \\ B(x,D)v^*(x,t)|_s = g(x,t), \end{cases} \tag{63}$$

In (63), introduce the change of variables $t = \tau^{\frac{1}{2}}$ then it becomes:

$$\begin{cases} 4\tau v_{\tau\tau}^* + 2v_\tau^* = P(x,D)v^*(x,\tau^{\frac{1}{2}}) + \tau^{\frac{1}{2}}P(x,D)\phi(x) \\ v^*(x,0) = 0 \qquad \lim_{\tau\to 0} v_\tau^*(x,\tau^{\frac{1}{2}}) = 0 \\ B(x,D)v^*(x,\tau^{\frac{1}{2}})|_s = g(x,\tau^{\frac{1}{2}}) \end{cases} \tag{64}$$

Now introduce the Laplace transform in (64) by transforming on the variable τ with transformed variable s. Then $\bar{v}^*(x,s)$ the Laplace transform of $v^*(x,\tau^{\frac{1}{2}})$, satisfies the problem

$$\begin{cases} 4s^2\frac{\partial}{\partial s}\bar{v}^*(x,s) + 6s\bar{v}^*(x,s) + P(x,D)\bar{v}^*(x,s) + \frac{\Gamma(\frac{3}{2})}{s^{\frac{3}{2}}}P(x,D)\phi(x) = 0 \\ B(x,D)\bar{v}^*(x,s)|_s = \bar{g}(x,s) \end{cases} \tag{65}$$

with $\bar{g}(x,s)$ the Laplace transform of $g(x,\tau^{\frac{1}{2}})$. Finally, a multiplication of the equation and conditions in (65) by $\frac{s^{\frac{3}{2}}}{\Gamma(\frac{3}{2})}$ leads to the problem

$$\begin{cases} 4s^2\frac{\partial}{\partial s}\left\{\frac{s^{\frac{3}{2}}\bar{v}^*}{\Gamma(\frac{3}{2})}\right\} + P(x,D)\left\{\frac{s^{\frac{3}{2}}\bar{v}^*}{\Gamma(\frac{3}{2})}\right\} + P(x,D)\phi(x) = 0 \\ B(x,D)\left\{\frac{s^{\frac{3}{2}}}{\Gamma(\frac{3}{2})}\bar{v}^*(x,s)\right\}|_s = \frac{s^{\frac{3}{2}}}{\Gamma(\frac{3}{2})}\bar{g}(x,s) \end{cases} \tag{66}$$

In (62), introduce the change of variables $t = \frac{1}{4s}$ for $s > 0$. Then (62) transforms into the problem

$$\begin{cases} 4s^2\frac{\partial}{\partial s}u^*\left(x,\frac{1}{4s}\right) + P(x,D)u^*\left(x,\frac{1}{4s}\right) + P(x,D)\phi(x) = 0 \\ B(x,D)u^*\left(x,\frac{1}{4s}\right)|_s = f\left(x,\frac{1}{4s}\right) \end{cases} \tag{67}$$

with

$$\lim_{s\to\infty} u^*\left(x,\frac{1}{4s}\right) = 0$$

A comparison of (66) and (67) shows that the functions $u^*(x,\frac{1}{4s})$ and $s^{\frac{3}{2}}(\bar{v}^*(x,s)/\Gamma\left(\frac{3}{2}\right))$ satisfy, firstly the same differential equation and secondly the same boundary conditions provided that

$$\begin{aligned} &(a) \quad f(x,\frac{1}{4s}) = \frac{s^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)}\bar{g}(x,s) \\ &(b) \lim_{s\to\infty} s^{\frac{3}{2}}\bar{v}^*(x,s) = 0 \end{aligned} \tag{68}$$

The conditions (68 a) are those covered by the hypotheses (58) and (60). Imposing these conditions along with (68 b), we get

$$\bar{v}^*(x,s) = \Gamma\left(\frac{3}{2}\right)s^{-\frac{3}{2}}u^*\left(x,\frac{1}{4s}\right) \tag{69}$$

and the result (61) follows by inversion and our definitions of u^* and v^*. The result (59) also follows from (69). This completes the proof. □

4 Explicit Solutions

In this section, we'll make good on our promises to solve problems (1) and (3) using transmutation methods. We intend to solve problem (1) first, and then demonstrate that the solution is correct by proving that the solution to problem (3) computed using the transmutation method is identical to the one computed using Fourier transforms. First, let us provide an explicit solution to the following degenerate hyperbolic equation:

$$\begin{cases} \partial_{tt}u - \Delta_x u - \langle x, \nabla_y u\rangle = 0\ , & \mathbb{R}^n_x \times \mathbb{R}^n_y \times (0,\infty) \\ u(x,y,0) = 0\ , & \partial_t u(x,y,0) = \psi(x,y) \end{cases}$$

using transmutations methods. By theorem (3.1) and using the solution of kolmogorov equation (55), we can conclude that if $f(x,y,t)$ is the solution of kolmogrov problem then

$$u(x,y,t) = \Gamma\left(\frac{3}{2}\right) L_s^{-1}\left\{s^{-\frac{3}{2}} f\left(x,y,\frac{1}{4s}\right)\right\}_{s\to t^2} \tag{70}$$

where $f(x,y,t)$ can be written as follows based on the computation presented in the (2.2):

$$f(x,y,t) = \frac{3^{\frac{n}{2}} t^{-2n}}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{-\frac{1}{4t}\left(|x-\bar{x}|^2 + 12\left|\frac{y-\bar{y}}{t} + \frac{x+\bar{x}}{2}\right|^2\right)} \psi(\bar{x},\bar{y}) d\bar{x} d\bar{y} \tag{71}$$

According to the linearity of Laplace inverse transform and Fobini theorem, we can simplify (70) as follows:

$$u(x,y,t) = \Gamma\left(\frac{3}{2}\right) L_s^{-1}\left\{s^{-\frac{3}{2}} \frac{3^{\frac{n}{2}}(4s)^{2n}}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{-s\left(|x-\bar{x}|^2 + 12\left|4s(y-\bar{y}) + \frac{x+\bar{x}}{2}\right|^2\right)} \psi(\bar{x},\bar{y}) d\bar{x} d\bar{y}\right\}_{s\to t^2}$$

change of variable as $x - \bar{x} = \xi$, $\quad y - \bar{y} = \eta$

$$= \frac{\sqrt{\pi}}{2} \frac{3^{\frac{n}{2}}(2)^{3n}}{\pi^n} L_s^{-1}\left\{s^{-\frac{3}{2}} \int_{\mathbb{R}^{2n}} s^{2n} e^{-s\left(|\xi|^2 + 12\left|4s\eta + \frac{2x-\xi}{2}\right|^2\right)} \psi(\xi - x, \eta - y) d\xi d\eta\right\}_{s\to t^2}$$

$$= \frac{\sqrt{\pi}}{2} \frac{3^{\frac{n}{2}}(2)^{3n}}{\pi^n} \int_{\mathbb{R}^{2n}} L_s^{-1}\left\{s^{-\frac{3}{2}} s^{2n} e^{-s\left(|\xi|^2 + 12\left|4s\eta + \frac{2x-\xi}{2}\right|^2\right)}\right\}_{s\to t^2} \psi(\xi - x, \eta - y) d\xi d\eta$$

Now we should compute the Laplace inverse part as follows:

$$L_s^{-1}\left\{s^{-\frac{3}{2}} s^{2n} e^{-s\left(|\xi|^2 + 12\left|4s\eta + \frac{2x-\xi}{2}\right|^2\right)}\right\}_{s\to t^2} = L_s^{-1}\left\{s^{-\frac{1}{2}} s^{2n-1} e^{-s\left(|\xi|^2 + 12\left|4s\eta + \frac{2x-\xi}{2}\right|^2\right)}\right\}_{s\to t^2}$$

$$= (2n-1)! \frac{\sqrt{\pi}}{t^{4n+1}} \delta(t^2 - \xi^2) * L_s^{-1}\left\{e^{-s\left(|\xi|^2 + 12\left|4s\eta + \frac{2x-\xi}{2}\right|^2\right)}\right\}_{s\to t^2}$$

$$= (2n-1)! \frac{\sqrt{\pi}}{t^{4n+1}} \delta(t^2 - \xi^2) * \frac{e^{\left(\frac{2x-\xi}{2}\right)^2}}{t^2 - 4\eta(2x-\xi)} * \frac{1}{t - 192\eta^2}$$

In the following theorem, we can present the final solution to our main problem (1).

Theorem 4.1. *Suppose we have the following degenerate hyperbolic partial differential equation:*

$$\begin{cases} \partial_{tt}u - \Delta_x u - \langle x, \nabla_y u\rangle = 0\ , & \mathbb{R}^n_x \times \mathbb{R}^n_y \times (0,\infty) \\ u(x,y,0) = 0\ , & \partial_t u(x,y,0) = \psi(x,y) \end{cases} \tag{72}$$

The following function satisfies this PDE equation:

$$u(x,y,t) = \frac{3^{\frac{n}{2}}(2)^{3n-1}}{t^{4n+1}\pi^{n-1}} \int_{\mathbb{R}^{2n}} Z(\xi,\eta,x,t)\psi(\xi - x, \eta - y) d\xi d\eta \tag{73}$$

where

$$Z(\xi,\eta,x,t) = (2n-1)!\ \delta(t^2 - \xi^2) * \frac{e^{\left(\frac{2x-\xi}{2}\right)^2}}{t^2 - 4\eta(2x-\xi)} * \frac{1}{t - 192\eta^2} \tag{74}$$

We now want to use the solution of the heat equation and transmutation methods to compute the solution of the wave equation.

Proposition 4.2. ***From solution of Heat equation to solution of Wave equation***
Suppose we have a solution to the Cauchy problem of the Heat equation:

$$\begin{cases} \partial_t u - \Delta_x u = 0\ , & \mathbb{R}^n \times (0,\infty) \\ u(x,0) = \psi(x) \end{cases} \tag{75}$$

Then the solution to the following wave equation:

$$\begin{cases} \partial_{tt} f - \Delta_x f = 0\ , & \mathbb{R}^n \times (0,\infty) \\ f(x,0) = 0\ , & \partial_t f(x,0) = \psi(x) \end{cases} \tag{76}$$

Is computed using:

$$f(x,t) = \Gamma\left(\frac{3}{2}\right) L_s^{-1} \left\{ s^{-\frac{3}{2}} u\left(x, \frac{1}{4s}\right) \right\}_{s \to t^2} \tag{77}$$

Proof. Consider the following fundamental solution of heat equation 75:

$$u(x,t) = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} \psi(y) dy \tag{78}$$

By theorem (3.1) and using the solution of heat equation (78), we can conclude that

$$f(x,t) = \Gamma\left(\frac{3}{2}\right) L_s^{-1} \left\{ s^{-\frac{3}{2}} u\left(x, \frac{1}{4s}\right) \right\}_{s \to t^2} \tag{79}$$

Let's keep expanding the computations until we get the form (32), which verifies our claim that the transmutation method is a correct method for solving a PDE. If we subsitute the heat equation solution (75), we obtain:

$$
\begin{aligned}
f(x,t) = \Gamma\left(\frac{3}{2}\right) L_s^{-1}\left\{s^{-\frac{3}{2}}\left(\frac{\pi}{s}\right)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-s|x-y|^2}\psi(y)dy\right\}_{s\to t^2} = \\
\frac{1}{2\pi}L_s^{-1}\left\{s^{-\frac{3}{2}}s^{\frac{3}{2}}\int_{\mathbb{R}^n} e^{-s|x-y|^2}\psi(y)dy\right\}_{s\to t^2} = \\
\frac{1}{2\pi}L_s^{-1}\left\{\int_{\mathbb{R}^n} e^{-s|x-y|^2}\psi(y)dy\right\}_{s\to t^2} = \\
\frac{1}{2\pi}L_s^{-1}\left\{\lim_{r\to\infty}\int_{B(x,r)} e^{-s|x-y|^2}\psi(y)dy\right\}_{s\to t^2}
\end{aligned}
$$

Where $B(x,r) = \{y \in \mathbb{R}^n \text{ s.t } |x-y| \leq r\}$. According to Cavalieri's principle

$$
\int_{B(x,r)} g(x)dx = \int_0^r \int_{S(x,r)} g(y)d\sigma(y)dr \tag{80}
$$

We can now proceed with the calculations using this principle:

$$
\begin{aligned}
f(x,t) = \frac{1}{2\pi}L_s^{-1}\left\{\lim_{r\to\infty}\int_0^r\int_{S(x,r)} e^{-sr^2}\psi(y)d\sigma(y)dr\right\}_{s\to t^2} \\
\frac{1}{2\pi}L_s^{-1}\left\{\lim_{r\to\infty}\int_0^r e^{-sr^2}\int_{S(x,r)}\psi(y)d\sigma(y)dr\right\}_{s\to t^2}
\end{aligned}
$$

According to the definition of spherical average of a function (29), we can compute the second integral as follows:

$$
\begin{aligned}
f(x,t) = \frac{1}{2\pi}L_s^{-1}\left\{\lim_{r\to\infty}\int_0^r e^{-sr^2}\left(4\pi r^2 M_\psi(x,r)\right)dr\right\}_{s\to t^2} = \\
2\int_0^\infty L_s^{-1}\left\{e^{-sr^2}\right\}_{s\to t^2} r^2 M_\psi(x,r)dr
\end{aligned}
$$

We can compute the inverse laplace inverse as follows:

$$
\begin{aligned}
f(x,t) = 2\int_0^\infty \delta\left(t^2-r^2\right)r^2 M_\psi(x,r)dr = \\
2\int_0^\infty \frac{1}{2r}\left[\delta\left(t-r\right)+\delta\left(t+r\right)\right]r^2 M_\psi(x,r)dr = \\
\int_0^\infty \left(rM_\psi(x,r)\right)\delta\left(t-r\right)dr + \int_0^\infty \left(rM_\psi(x,r)\right)\delta\left(t+r\right)dr = \\
\int_0^\infty \left(rM_\psi(x,r)\right)\delta\left(t-r\right)dr + \int_{-\infty}^0 \left(zM_\psi(x,-z)\right)\delta\left(t-z\right)dr
\end{aligned}
$$

Since spherical average function is spherically symmetric, we have

$$f(x,t) = \int_0^\infty (rM_\psi(x,r))\,\delta(t-r)\,dr + \int_{-\infty}^0 (zM_\psi(x,z))\,\delta(t-z)\,dr =$$
$$\int_0^\infty (rM_\psi(x,r))\,\delta(t-r)\,dr + \int_{-\infty}^0 (rM_\psi(x,r))\,\delta(t-r)\,dr =$$
$$\int_{-\infty}^\infty (rM_\psi(x,r))\,\delta(t-r)\,dr$$

Using the following Dirac delta property:

$$\int_{-\infty}^\infty g(y)\delta(x-y)\,dy = g(x) \tag{81}$$

Finnally, we achieve the form (32) for when $\varphi(x) = 0$,

$$f(x,t) = \int_{-\infty}^\infty (rM_\psi(x,r))\,\delta(t-r)\,dr = tM_\psi(x,t)$$

This concludes our claim. □

5 Conclusion

Finally, we learned how to use a strategy in this paper to divide a difficult problem into two simpler ones and solve them using an equivalent problem of lower degree in terms of time. These methods can aid in the solution of previously unsolved PDEs. The explicit solution computed in this article can also address a number of other issues related to gas kinetic theory. For future studies, we recommend developing an analogous numerical method based on the transmutation technique with optimal error margins.

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