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Research paper

Robustness in Mean-Variance Portfolio Optimization

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Abstract:
In this paper, we discuss some of the concepts of robustness for uncertain multi-objective optimization problems. An important factor involved with multi-objective optimization problems is uncertainty. The uncertainty may arise from the estimation of parameters in the model, error of computation, the structure of a problem, and so on. Indeed, some parameters are often unknown at the beginning of solving a multi-objective optimization problem. One of the most important and popular approaches for dealing with uncertainty is robust optimization. Markowitz's portfolio optimization problem is strongly sensitive to the perturbations of input parameters. We consider Markowitz's portfolio optimization problem with ellipsoid uncertainty set and apply set-based minmax and lower robust efficiency to this problem. The concepts of robust efficiency are used in the real stock market and compared to each other. Finally, the increase and decrease effects of uncertainty set parameters on these robust efficient solutions are verified.

Keywords: Portfolio Optimization, Robustness, Ellipsoid Uncertainty Set.
MSC Classification: 91G10, 90C26, 90C31

1 Introduction

Multiobjective (vector) optimization is concerned with optimizing more than one objective function subject to some constraints. In multiobjective optimization, the decision maker is faced with a set of conflicting criteria and the goal is to choose the most preferred alternative(s). Many real-life optimization problems are multi-objective in their nature. In finance, we commonly deal with two objectives, maximizing return (profit) and minimizing risk. In many classes of practical optimization problems, a decision maker is faced with uncertainty in the modeling, resulting from data errors, perturbations, data uncertainty, environmental factors, and partial knowledge. There are various approaches in the literature for dealing with uncertainty, including stochastic optimization, robust optimization, and sensitivity analysis. We consider Markowitz's portfolio optimization problem that heavily suffers from uncertainties of input parameters [11, 12]. Robust optimization has been widely studied in recent years from different standpoints in single-

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and multi-objective optimization, see for example [1, 2, 8, 15]. Soyster [15] initially introduced minmax robustness in single-objective optimization, and then, Ben-Tal et al. [1] and Ben-Tal and Nemirovski [2] studied it extensively. This notion was extended from single-objective to multi-objective optimization by Ehrgott et al. [6], Fliege and Werner [7], Bokrantz and Fredriksson [3], and Kuroiwa and Lee [10] . The concept of robustness in portfolio optimization problems has been used by several researchers. Fliege and Werner used minmax robust efficiency in Markowitz's portfolio optimization problem and set order relations approach employed in the portfolio optimization problem [4, 7].

In this paper, we use robust efficiency in the sense of Ehrgott and lower set less ordered efficiency in a portfolio optimization problem with ellipsoid uncertainty set. In the following, we apply these concepts to 9 investment positions from the Tehran stock market and compared them to each other.

The rest of this paper is organized as follows. Section 2 contains some preliminaries. Section 3 is devoted to introducing some robustness concepts. Section 4 continues the paper with robust portfolio optimization. Section 5 illustrates robust efficient solutions for portfolios in the real stock market finally, Section 6 draws some conclusions.

2 Preliminaries

This section is devoted to some preliminaries. For two arbitrary sets $A, B \subseteq \mathbb{R}^n$, we use the notation $A \pm B := \{x \pm y : x \in A,\ y \in B\}$. For simplicity, we use $x \pm A$ instead of $\{x\} \pm A$.

A set $K \subseteq \mathbb{R}^p$ is said to be a cone if $\lambda K \subseteq K$ for each $\lambda \in [0, +\infty)$. The cone K is said to be convex if $K + K \subseteq K$, and it is pointed if $K \cap (-K) = \{0\}$.Furthermore, it is called nontrivial if $K \neq \mathbb{R}^n$ and $K \neq \{0\}$. A set $K \subseteq \mathbb{R}^m$ is called an ordering cone if it is a nontrivial, closed, convex, and pointed cone; which establishes a partial ordering in $\mathbb{R}^p$. The natural ordering cone is defined by

$$\mathbb{R}^p_{\geqq} = \{x = (x_1, \cdots, x_p) \in \mathbb{R}^p \ :\ x_i \geq 0,\ i = 1, \cdots p\}.$$

Throughout the paper, the notations $\leqq$, $\leq$ and $<$ stand for the following orderings on $\mathbb{R}^p$:

$$x \leqq y \iff y - x \in \mathbb{R}^p_{\geqq},$$

$$x \leq y \iff y - x \in \mathbb{R}^p_{\geq},$$

$$x < y \iff y - x \in \mathbb{R}^p_{>},$$

where $\mathbb{R}^p_{>} = \{x \in \mathbb{R}^p \ :\ x_i > 0,\ i = 1, \cdots p\}$ and the symbol $\mathbb{R}^p_{\geq}$ denotes the set $\mathbb{R}^p_{\geqq} \setminus \{0\}$.

Throughout the paper, the notation $\|.\|$ stands for Euclidean norm and $\langle a.b \rangle$ and $a^T b$ denotes the inner product of $a, b \in \mathbb{R}^n$. In matrix spaces the notation $\|\Sigma\|$

stands for Frobenius norm of $\Sigma \in \mathbb{S}^n$, where $\mathbb{S}^n$ is the space of all symmetric $n \times n$ matrices and $\mathbb{S}^n_+$ is the cone of positive semidefinite matrices.

2.1 Multi-objective Optimization

We consider the following multi-objective optimization problem:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in X. \end{array} \tag{1}$$

where $\emptyset \neq X \subseteq \mathbb{R}^n$ is the feasible solution set of this problem and $f : X \to \mathbb{R}^p$ with $p \geq 2$. In fact $f(x) = (f_1(x), f_2(x), \cdots, f_p(x))$ for each x.
This class of multiobjective optimization problems is called a deterministic multi-objective optimization problem (DMOP) without uncertain parameters.

Definition 2.1. [5] A vector $\bar{x} \in X$ is called an efficient solution of (DMOP) if

$$\Big(f(\bar{x}) - K\Big) \bigcap f(X) = \{f(\bar{x})\}.$$

Utilizing a natural ordering cone, a vector $\bar{x} \in X$ is called an efficient solution of (DMOP) if there exists no $x \in X$ such that $f(x) \leq f(\bar{x})$ and $f(x) \neq f(\bar{x})$.

2.2 Scalarization

Scalarization is a traditional approach to solving multi-objective optimization problems [5]. By scalarization methods, one solves a single objective optimization problem corresponding to a given multiobjective optimization problem whose optimal solutions can be efficient.

- **Weighted Sum Scalarization**
  The single objective weighted sum problem corresponding to (MOP) is as follows
  $$\begin{array}{lll} (P_\lambda): & \min & \sum\limits_{i=1}^{p} \lambda_i f_i(x) \\ & \text{s.t.} & x \in X \end{array}$$
  where $\lambda_i \geq 0$ for $i = 1, \cdots, p$ and $\sum_{i=1}^{p} \lambda_i = 1$.
- **The ε-Constraint Method**
  Besides the weighted sum approach, the ε-constraint method is probably the best-known technique to solve multicriteria optimization problems.
  $$\begin{array}{lll} (P_\varepsilon): & \min & f_j(x) \\ & \text{s.t.} & x \in X \\ & f_k(x) \leq \varepsilon_k & k = 1, \cdots, p,\ k \neq j, \end{array}$$
  where $\varepsilon \in \mathbb{R}^p$.

2.3 Mean-Variance Portfolio Optimization Problem

Consider a portfolio of n risky assets (A_i). Markowitz mean-variance portfolio optimization problem (MV) can be written as

$$\begin{aligned} (MV): \min \quad & f(x) = (x^T\hat{\Sigma}x, -\hat{\mu}x) \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i = 1 \\ & m_i \leq x_i \leq M_i, \quad i = 1, \cdots, n. \end{aligned} \tag{2}$$

In 2, $x^T\hat{\Sigma}x$ and $-\hat{\mu}x$ are measuring the risk and return of portfolio respectively. The parameters of $\hat{\mu}$ and $\hat{\Sigma}$ are assumed to be accurately computed by

$$\hat{\mu_i} = E[R_i],$$

$$\hat{\Sigma}_{ij} = E[(R_i - \hat{\mu_i})(R_j - \hat{\mu_j})],$$

where R_i is the return on A_i, $i = 1, \cdots, n$, M_i and m_i are the maximum and minimum investments on A_i respectively.
The basic idea is that a portfolio is solely characterized by the two quantities risk (mostly measured in terms of the variance or volatility) and expected return. Since an investor is seeking an allocation with low risk and high return, a trade-off between these two conflicting aims has to be made. In this paper, we solve the UMV problem with weighted sum scalarization, therefore we are facing the following problem

$$\begin{aligned} (MV_\lambda): \min \quad & (1-\lambda)(x^T\hat{\Sigma}x) + \lambda(-\hat{\mu}x) \\ \text{s.t.} \quad & x \in X \end{aligned}$$

where $0 \leq \lambda \leq 1, X := \{x \in \mathbb{R}^n_+ : \sum_{i=1}^{N} x_i = 1,\ m_i \leq x_i \leq M_i, \quad i = 1, \cdots, n\}$.

2.4 Uncertain Multi-objective Optimization

The robust optimization approach finds solution(s)while uncertainty is involved in objective function and constraints.

The following vector programming problem is a perturbed multi-objective optimization problem whose modeling is due to a perturbation in the objective function:

$$\begin{aligned} P(\xi): min \quad & f(x, \xi) \\ \text{s.t.} \quad & x \in X, \xi \in U. \end{aligned} \tag{3}$$

where x and ξ are decision variables and uncertain parameters respectively, $f : \mathbb{R}^n \times U \to \mathbb{R}^p$ and $U \subseteq R^q$. Uncertainty can emerge anywhere in the real world (e.g., the preceding expected returns possibly varying in a set of scenarios called an uncertainty set ($U \subseteq \mathbb{R}^p$)). Therefore an uncertain the multi-objective optimization problem (UMOP) is written as

$$P(U) := (P(\xi), \xi \in U)$$

defined as a family of parametric problems $P(\xi)$.

In uncertain Markowitz mean-variance portfolio optimization problem (UMV), the parameters of the expected return and the covariance matrix are considered to be uncertain. For this purpose a joint uncertainty set U is chosen for the uncertain data $\xi = (\mu, \Sigma)$.

$$\begin{aligned} (UMV): \min_{\xi=(\mu,\Sigma)\in U} \quad & f(x) = (x^T \Sigma x, -\mu x) \\ \text{s.t.} \quad & \sum_{i=1}^{N} x_i = 1 \\ m_i \leq x_i \leq M_i, \quad & i = 1, \cdots, n. \end{aligned} \tag{4}$$

In the existing literature, there are some uncertainty sets that are employed in uncertain optimization problems [14].
In this paper, we choose an ellipsoid uncertainty set [14] around the $(\hat{\mu}, \hat{\Sigma})$:

$$U_\delta(\hat{\mu}, \hat{\Sigma}) := \{(\mu, \Sigma) \in \mathbb{R}^n \times \mathbb{S}^n_+ \mid \|\mu - \hat{\mu}\| + c\|\Sigma - \hat{\Sigma}\| \leq \delta\}.$$

The point $(\hat{\mu}, \hat{\Sigma})$) is called the center of the ellipsoid set and δ is its radius.

3 Robust Portfolio Optimization

One of the most important and popular approaches for dealing with uncertainty is robust optimization and an important issue in robust optimization the approach is studying the efficient solutions which are insensitive to some changes in the problem data. As mentioned in the preceding section, various definitions of robustness can be found in the literature for example [6–8].

Definition 3.1. [7] A vector $\bar{x} \in X$ is called robust efficient in the sense of Fliege and Werner for $P(U)$ (point-based minmax robust), written as $\bar{x} \in FWR[f, U, X]$, if $\bar{x}$ is efficient solution for

$$\min_{x \in X} \max_{\xi \in U} f(x, \xi) = \min_{x \in X} (\max_{\xi \in U} f_1(x, \xi), \cdots, \max_{\xi \in U} f_p(x, \xi)).$$

This means that the worst case of the objective function under all possible scenarios are minimized.

Definition 3.2. [6] A vector $\bar{x} \in X$ is called robust efficient in the sense of Ehrgott for $P(U)$ (set-based minmax robust), written as $\bar{x} \in ER[f, U, X]$, if there is no $x \in X \setminus \{\bar{x}\}$ such that

$$f_U(x) \subseteq f_U(\bar{x}) - \mathbb{R}^p_{\geq},$$

where $f_U(x) := \{f(x, \xi) : \xi \in U\}$.

Ehrgott et al. [6] pointed out that if $U = U1 \times \cdots \times U_p$ and the objective functions $f_1, \cdots, f_p$ are independent of each other with respect to the uncertain set, then point-based minmax robust efficiency is equivalent to set-based minmax robust efficiency.

Definition 3.3. [8] A vector $\bar{x} \in X$ is called a lower set less ordered efficient solution for $P(U)$, written as $\bar{x} \in LSO[f, U, X]$, if there is no $x \in X \setminus \{\bar{x}$ such that

$$f_U(x) + \mathbb{R}^p_{\geq} \supseteq f_U(\bar{x}).$$

Now, we apply the concepts presented in the previous on the UMV problem and compare them. In the UMV problem, it is evident that set-based minmax robust efficiency is equivalent to point-based robust efficiency [6, Theorem 4.1].

The robust counterpart in the sense of Ehrgott of UMV problem [7] with ellipsoid uncertainty set can be given as

$$\begin{aligned}(UMV(ER)): \quad \min_{x\in X} \max_{\xi=(\mu,\Sigma)\in U} f(x,\xi) &= \min_{x\in X} \left(\max_{(\mu,\Sigma)\in U_\delta(\hat{\mu},\hat{\Sigma})} x^T\Sigma x \ , \max_{(\mu,\Sigma)\in U_\delta(\hat{\mu},\hat{\Sigma})} -\mu^T x \right) \\ &= \min_{x\in X} \left(x^t\hat{\Sigma}x + \frac{\delta}{c}\|x\|^2 \ , -\hat{\mu}^t x + \delta\|x\| \right)\end{aligned}$$

this relationship tells the fact of computing upper robust efficient solutions for UMV portfolio optimization problems via MV_{minmax}.

According to definition 3.3 the lower robust efficiency of UMV problem with ellipsoid uncertainty set is obtained as follows

$$\begin{aligned}(UMV(LSO)): \quad \min_{x\in X} \min_{\xi=(\mu,\Sigma)\in U} f(x,\xi) &= \min_{x\in X} \left(\min_{(\mu,\Sigma)\in U_\delta(\hat{\mu},\hat{\Sigma})} x^T\Sigma x \ , \min_{(\mu,\Sigma)\in U_\delta(\hat{\mu},\hat{\Sigma})} -\mu^T x \right) \\ &= \min_{x\in X} \left(x^t\hat{\Sigma}x - \frac{\delta}{c}\|x\|^2 \ , -\hat{\mu}^t x - \delta\|x\| \right)\end{aligned}$$

Unlike to UMV(ER), UMV(LSO) employs to minimize the function with uncertain parameter ξ taking the value at the best case (the best scenario) and the optimization variable x fixed while reducing uncertainty in the optimization problem. And the efficient solutions obtained by UMV(LSO) are lower robust efficient solutions.

So, with the help of the weighted sum scalarization, the above two problems become as follows

$$(UMV_\lambda(ER)): \quad \min_{\substack{x\in X \\ 0\leq\lambda\leq 1}} (1-\lambda)(x^t\hat{\Sigma}x + \frac{\delta}{c}\|x\|^2) + \lambda(-\hat{\mu}^t x + \delta\|x\|),$$

and

$$(UMV_\lambda(LSO)): \quad \min_{\substack{x\in X \\ 0\leq\lambda\leq 1}} (1-\lambda)(x^t\hat{\Sigma}x - \frac{\delta}{c}\|x\|^2) + \lambda(-\hat{\mu}^t x - \delta\|x\|).$$

4 Illustrations of robust portfolio solutions

In this section, we will compare the set-based minmax and the lower robust efficiency in the UMV problem via the data from the real stock market. We have considered 9 investment positions from the Tehran stock market in a period of 3 years (1998/01/17 to 1401/04/12). The information related to the daily price of these 9 investment positions in this period of time has been obtained with the help of TseClient 2.0 software, then returns and volatilities of these assets have been calculated using MATLAB software. Expected return, volatilities of these assets, and the covariance between them can be seen in Table 1 and Table 2.

Name	Bank Melli Inv	S*Mellat Bank	Tamin Daroo	Gol-E-Gohar	Iran Khodro	Shazand Petr	S*Isf. Oil Ref.Co	Sadr Tamin Inv	Iran Const. Inv
Symbol	BANK1	BMLT1	DTIP1	GOLG1	IKCO1	PARK1	PNES1	SADR1	SAKH1
$\hat{\mu}$	0.0013	-0.0002	0.0015	0.0007	-0.0007	0.0022	-0.0004	0.0030	-0.0014
σ	0.0012	0.0038	0.0029	0.0021	0.0131	0.0009	0.0029	0.0027	0.0155

Table 1: Expected return, Volatilities of assets

Name	Bank Melli Inv	S*Mellat Bank	Tamin Daroo	Gol-E-Gohar	Iran Khodro	Shazand Petr	S*Isf. Oil Ref.Co	Sadr Tamin Inv	Iran Const. Inv
Symbol	BANK1	BMLT1	DTIP1	GOLG1	IKCO1	PARK1	PNES1	SADR1	SAKH1
BANK1	0.0012	0.0003	0.0003	0.0003	0.0002	0.0004	0.0004	0.0003	0.0003
BMLT1	0.0003	0.0038	0.0004	0.0003	0.0002	0.0003	0.0002	0.0005	0.0001
DTIP1	0.0003	0.0004	0.0029	0.0002	0.0001	0.0002	0.0002	0.0003	0.0003
GOLG1	0.0003	0.0003	0.0002	0.0021	0.0005	0.0003	0.0004	0.0003	0.0002
IKCO1	0.0002	0.0002	0.0001	0.0005	0.0131	0.0004	0.0007	0.0006	0.0002
PARK1	0.0004	0.0003	0.0002	0.0003	0.0004	0.0009	0.0004	0.0004	0.0001
PNES1	0.0004	0.0002	0.0002	0.0004	0.0007	0.0004	0.0029	0.0003	0.0003
SADR1	0.0003	0.0005	0.0003	0.0003	0.0006	0.0004	0.0003	0.0027	0.0003
SAKH1	0.0003	0.0001	0.0003	0.0002	0.0002	0.0001	0.0003	0.0003	0.0155

Table 2: Covariance between assets

We obtained an efficient solution and set-based minmax and lower robust efficient solution of the UMV problem with an ellipsoid uncertainty set. At first, we take $c = 2$ and $\delta = 0.075$ as parameters of the ellipsoid uncertainty set. The results of MV, UMV(ER), and UMV(LSO) are illustrated in Fig. 2. Then, the cases of $\delta = 0.0075$ and $\delta = 0.00075$, and the value c unchanged are shown in Fig. 3 and 4. Oppositely, the cases of $c = 0.2$ and $c = 20$ and δ unchanged are depicted in Fig. 5 and 6 [7].

Comparisons of Figs. 2, 3, and 4 show that the radius of ellipsoid uncertainty set (δ) has a positive effect on the set-based minmax efficient (robust efficient in sense of Ehrgott) frontier. That is, with the decrease of the radius, the set-based minmax efficient frontier will be closer to the efficient frontier. On the other, a lower robust efficient frontier is sensitive to the scaling factor of ellipsoid uncertainty set (c). Fig. 5 and 6 illustrated that the increase of the c causes the lower robust efficient frontier to be closer to the efficient frontier.

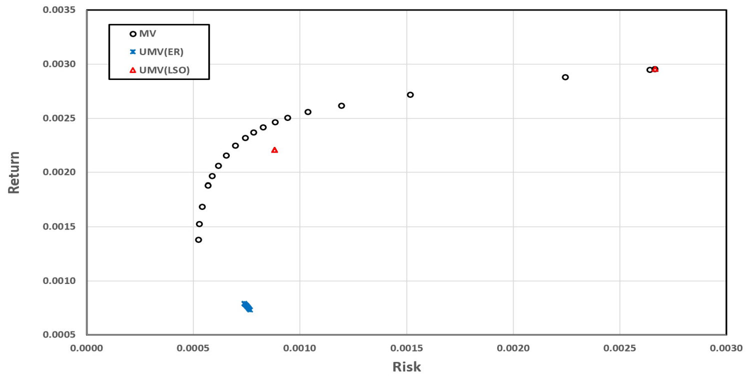

Figure 1: Efficient frontier when $c = 2$, $\delta = 0.075$

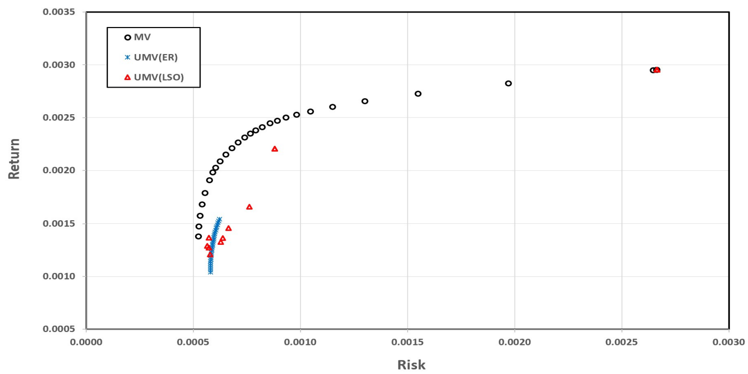

Figure 2: Efficient frontier when $c = 2$, $\delta = 0.0075$

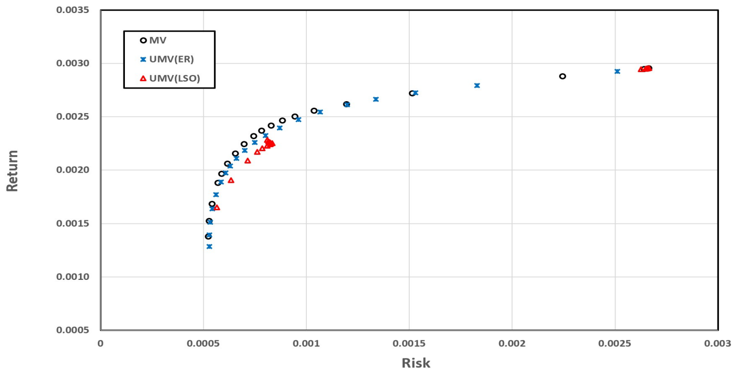

Figure 3: Efficient frontier when $c = 2$, $\delta = 0.00075$

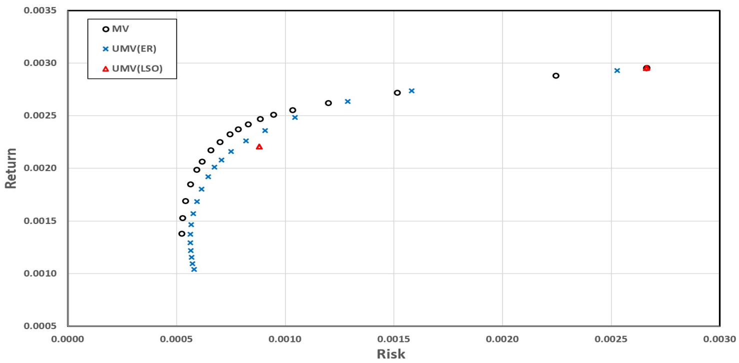

Figure 4: Efficient frontier when $c = 0.2$, $\delta = 0.00075$

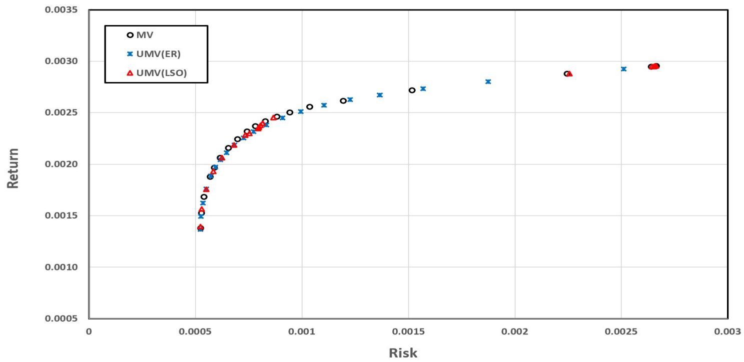

Figure 5: Efficient frontier when $c = 20$, $\delta = 0.00075$

5 Conclusion

In this paper, some concepts of robustness for uncertain multi-objective optimization problems are introduced and set-based minmax robust and lower robust efficiency are applied in the Markowitz's portfolio optimization problem with ellipsoid uncertainty set. In the end, we applied these concepts to the real market and investigated the effects of uncertainty set parameters on these robust efficient solutions. These outcomes explain that in the real stock market, each of the (robust) efficient solutions can be recommended to construct the advisable portfolio according to the status of the stock market.

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