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The Beta Modified Exponential Power Series Distribution: Properties and Applications

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Abstract: We introduce the Beta Modified Exponential Power Series (BMEPS) distribution, a parametric model adept at handling increasing, decreasing, bathtub-shaped, and unimodal failure rates. Constructed to address a latent complementary risk problem, this distribution amalgamates elements from the Beta Modified Exponential (BME) and power series distributions. Notably, it encompasses essential distributions found in the literature, like the Beta Modified Exponential Poisson (BMEP), Beta Modified Exponential Geometric (BMEG), and Beta Modified Exponential Logarithmic (BMEL) models as special subtypes. This study includes a detailed mathematical treatment of the BMEPS distribution, providing closed-form expressions for its density, cumulative distribution, survival function, failure rate function, r-th raw moment, and order statistics moments. Additionally, we explore maximum likelihood estimation and present Fisher information matrix components. Lastly, we demonstrate the versatility of this distribution by applying it to real-world data.

Keywords: Beta modified exponential distribution, Power series distributions, Goodness-of-fit Tests, Hazard Function, Residual life function.

Mathematics Subject Classification (2010): 62F03.

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1. Introduction

The well-known exponential distribution has been widely studied and applied in various aspects. The assumption of a monotonic failure rate of a product, however, may not be suitable and accurate in reality. On the other hand, when modeling monotone hazard rates, the Exponential distribution may be an initial choice because of the possibility of its negatively and positively skewed density shapes. Several distributions have been proposed in the literature to model lifetime data by compounding some useful lifetime distributions. Kuş (2007) introduced a two-parameter distribution known as the exponential-Poisson (EP) distribution, which has a decreasing failure rate, by compounding an exponential distribution with a Poisson distribution. In the same fashion, the modified Weibull (MW) distribution was proposed by Lai et al. (2003) and the extended flexible Weibull distribution was given by Bebbington et al. (2007).

Lifetime distributions and power series distributions constitute fundamental components within statistical modeling, offering versatile frameworks for analyzing diverse sets of data across numerous fields. This study embarks on a comprehensive exploration by synthesizing findings from prominent contributions in this domain. In this concept, the Beta exponential (BE) distribution is introduced by Nadarajah and Kotz (2006), the exponential-geometric (EG) is introduced by Adamidis and Loukas (1998), the exponential-logarithmic (EL) is introduced by Tahmasbi and Rezaei (2008), the exponential-power series (EPS) is introduced by Chahkandi and Ganjali (2009), and the generalized exponential power series (GEPS) by Mahmoudi and Jafari (2012). Also, Barreto-Souza and Cribari-Neto (2009) and Louzada et al. (2011) introduced the exponentiated exponential-Poisson (EEP) and the complementary exponential-geometric (CEG) distributions where the EEP is the generalization of the EP distribution, and the CEG is complementary to the EG model proposed by Adamidis and Loukas (1998).

For more studies in this concept, Bagheri et al. (2016) presented the Generalized Modified Weibull Power Series Distribution, enriching the theoretical understanding and practical applications of this model. Oluyede et al. (2020) further expanded this discourse by shedding light on the Exponentiated Generalized Power Series distribution, elucidating its properties and wide-ranging applications. Raffiq et al. (2022) delved into the Marshall–Olkin Inverted Nadarajah–Haghighi Distribution, offering insights into its estimation techniques and practical usage. Additionally, Osatohanmwen et al. (2022) contributed a novel perspective through the Exponentiated Gumbel–Weibull Logistic distribution, specifically applying it to model COVID-19 infections in Nigeria. Finally, Khojastehbakht et al. (2023) explored the Beta Exponential Power Series Distribution, augmenting the understanding of distributional characteristics within the realm of lifetime distributions and power series distributions. This amalgamation of studies collectively enriches the landscape of statistical distributions, fostering a deeper comprehension and wider application scope within this crucial domain of statistical analysis.

In this paper, we introduce the Beta Modified Exponential Power Series (BMEPS) distributions aiming to enhance statistical modeling. BMEPS distributions provide a comprehensive theoretical framework that includes their density, failure rate functions, and a detailed study of specific cases. These distributions are shown to handle various failure rate patterns—ranging from increasing and decreasing to bathtub-shaped and unimodal. In Section 2, we establish the novel category of Beta Modified Exponential Power Series (BMEPS) distributions, presenting their density and failure rate functions. Additionally, we delve into the detailed study of specific distributions within this framework, demonstrating that the failure rate can exhibit varying patterns. It may increase, decrease, assume a bathtub shape, or display a unimodal form. In Section 3, we derive quantiles and moments of BMEPS distributions, alongside elucidating the probability density function of the ith order statistic. Indeed, Section 3 furnishes expressions for the rth raw moments of the BMEPS distribution and the ith order statistic. The methodology for estimation via maximum likelihood and the expression for Fisher's information matrix are expounded in Section 4. Finally, in Section 5, we showcase the flexibility and potential of the new distribution through its application to a real dataset.

2. The class of BMEPS distribution

Marshall and Olkin (1997) introduced a parameterization scheme for a distribution function $F(y, a)$ by defining another distribution function as

$$F(y, a) = \frac{F(y)}{F(y) + a(1 - F(y))}, y \in R, a > 0.$$

We use the parameterization to obtain the two-parameter distribution known as the modified exponential (ME) distribution function

$$\hat{F}(y) = \frac{1 - e^{-\beta y}}{1 - (1 - \alpha)e^{-\beta y}}, y > 0, \alpha, \beta > 0.$$

Preda et al. (2011) introduced this distribution. We obtain the cumulative distribution function (CDF) of the BME distribution with five parameters $\alpha \geq 0$, $\beta \geq 0$, $a > 0$, $b > 0$ and $k > 0$ as

$$G(y) = I_{\hat{F}(y)}(\alpha, \beta), y > 0, a, b, \alpha, \beta > 0.$$

where $I_{F(y)}(\alpha,\beta)$ is

$$I_{F(y)}(\alpha,\beta)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^{F(y)} t^{a-1}(1-t)^{b-1}dt, y>0, a, b, \alpha, \beta>0.$$

and the density function of the BME distribution is given by

$$g(y)=(\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)})(\frac{1-e^{-\beta y}}{1-(1-\alpha)e^{-\beta y}})^{(a-1)}(\frac{\alpha e^{-\beta y}}{1-(1-\alpha)e^{-\beta y}})^{(b-1)}(\frac{\alpha\beta e^{-\beta y}}{(1-(1-\alpha)e^{-\beta y})^2}).$$

The significance of this distribution lies in its capability to model both monotonic and non-monotonic failure rates, prevalent in lifetime problems and reliability analyses. Suppose N represents a random variable denoting the count of failure causes, where $N=1,2,....$ If we consider N to follow a power series distribution (truncated at zero), its probability function is defined as follows

$$p(N=n)=\frac{a_n\theta^n}{C(\theta)},\ n=1,2,...\ \ and\ \ \theta\in(0,S),$$

where $a_1, a_2, ...$ is a sequence of nonnegative real numbers, where at least one of them is strictly positive, S is a positive number no greater than the ratio of convergence of the power series $\sum_{n=1}^{\infty} a_n\theta^n$ and $C(\theta)=\sum_{n=1}^{\infty} a_n\theta^n$. Useful quantities of some power series distributions are given in Table 1.

The Beta Modified Exponential Power Series Distribution, denoted by BMEPS $(\alpha,\beta,a,b,k,\theta)$, is defined by the marginal CDF of $X=\max\{Y_1,Y_2,\ldots,Y_N\}$, i.e.,

$$F(x)=1-\frac{C\Big(\theta(I_{F(x)}(\alpha,\beta))^k\Big)}{C(\theta)} \tag{2.1}$$

where $\alpha,\beta,\theta>0$ and $a,b,k\geq 0$. The PDF of the BMEPS$(\alpha,\beta,a,b,k,\theta)$ is given by

$$f(x)=\frac{B_{k,\theta}(\frac{1-e^{-\beta x}}{A_{\alpha,\beta}})^{(a-1)}(\frac{\alpha e^{-\beta x}}{A_{\alpha,\beta}})^{(b-1)}(\frac{\alpha\beta e^{-\beta x}}{(A_{\alpha,\beta})^2})(I_{F(x)}(\alpha,\beta))^{k-1}C'\Big(\theta\big(I_{F(x)}(\alpha,\beta)\big)^k\Big)}{C(\theta)},$$

where $A_{\alpha,\beta}=1-(1-\alpha)e^{-\beta x}$ and $B_{k,\theta}=k\theta(\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)})$. The survival function and hazard rate function of the BMEPS distribution are given, respectively, by

$$S(x)=\frac{C\Big(\theta\big(I_{F(x)}(\alpha,\beta)\big)^k\Big)}{C(\theta)},$$

and

$$h(x)=\frac{B_{k,\theta}(\frac{1-e^{-\beta x}}{A_{\alpha,\beta}})^{(a-1)}(\frac{\alpha e^{-\beta x}}{A_{\alpha,\beta}})^{(b-1)}(\frac{\alpha\beta e^{-\beta x}}{(A_{\alpha,\beta})^2})(I_{F(x)}(\alpha,\beta))^{k-1}C'\Big(\theta\big(I_{F(x)}(\alpha,\beta)\big)^k\Big)}{C\Big(\theta\big(I_{F(x)}(\alpha,\beta)\big)^k\Big)}.$$

The BMEPS distribution is a mixture of the BME distribution with weights defined by the power series distribution.

Table 1: Useful quantities of some power series distributions

Distribution	a_n	$C(\theta)$	$C^{'}(\theta)$	$C^{''}(\theta)$	$C(\theta)^{-1}$	$\ominus$
Poisson	$n!^{-1}$	$e^{\theta}-1$	e^{θ}	e^{θ}	$log(\theta+1)$	$\theta\in(0,\infty)$
Logarithmic	n^{-1}	$-log(1-\theta)$	$(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$1-e^{-\theta}$	$\theta\in(0,1)$
Geometric	1	$\theta(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$2(1-\theta)^{-3}$	$\theta(\theta+1)^{-1}$	$\theta\in(0,1)$
Binomial	$\binom{m}{n}$	$(\theta+1)^{m}-1$	$m(\theta+1)^{m-1}$	$\frac{m(m-1)}{(\theta+1)^{2-m}}$	$(\theta-1)^{\frac{1}{m}}-1$	$\theta\in(0,1)$

2.1 Special cases of BMEPS

In the following subsection we study in detail some special cases of the class BMEPS of distributions. This class of distributions contains several lifetime models such as: beta modified exponential Poisson (BMEP), beta modified exponential logarithmic (BMEL), and beta modified exponential geometric (BMEG) and distributions as special cases.

2.1.1 Beta Modified Exponential Poisson Distribution

The beta modified exponential Poisson (BMEP) distribution is a special case of the BMEPS distributions with $a_n=\frac{1}{n!}$ and $C(\theta)=e^{\theta}-1$. Using the CDF (2.1), the CDF of the Beta Modified Exponential Poisson (BMEP) distribution is given by

$$F_{BMEP}(x)=1-\frac{e^{\theta(I_{F(x)}(\alpha,\beta))^{k}}-1}{e^{\theta}-1}$$

where $\alpha,\beta,\theta>0$ and $a,b,k\geq 0$. The associated PDF and hazard rate function of this distribution are given, respectively, by

$$f_{BMEP}(x)=\frac{B_{k,\theta}(\frac{1-e^{-\beta x}}{A_{\alpha,\beta}})^{(a-1)}(\frac{\alpha e^{-\beta x}}{A_{\alpha,\beta}})^{(b-1)}(\frac{\alpha\beta e^{-\beta x}}{(A_{\alpha,\beta})^{2}})(I_{F(x)}(\alpha,\beta))^{k-1}e^{\theta(I_{F(x)}(\alpha,\beta))^{k}}}{e^{\theta}-1}$$

and

$$h_{BMEP}(x)=\frac{B_{k,\theta}(\frac{1-e^{-\beta x}}{A_{\alpha,\beta}})^{(a-1)}(\frac{\alpha e^{-\beta x}}{A_{\alpha,\beta}})^{(b-1)}(\frac{\alpha\beta e^{-\beta x}}{(A_{\alpha,\beta})^{2}})(I_{F(x)}(\alpha,\beta))^{k-1}e^{\theta(I_{F(x)}(\alpha,\beta))^{k}}}{e^{\left(\theta\left(I_{F(x)}(\alpha,\beta)\right)^{k}\right)}-1}.$$

The plots of the PDF, CDF and hazard rate function of the BMEP distribution for some values of α,β,θ,a,b and k are given in Figure 1. Models that present a bathtub-shaped failure rate are very useful in survival analysis. The modeling and analysis of lifetimes are important aspect of statistical work in a wide variety of scientific and technological fields. The new distribution due to its flexibility in accommodating all forms of the risk function seems to be an important distribution that can be used in a variety of problems in modeling survival data.

The exponential distribution does not provide a reasonable parametric fit for modeling phenomena with non-monotone failure rates such as the bathtub-shaped and

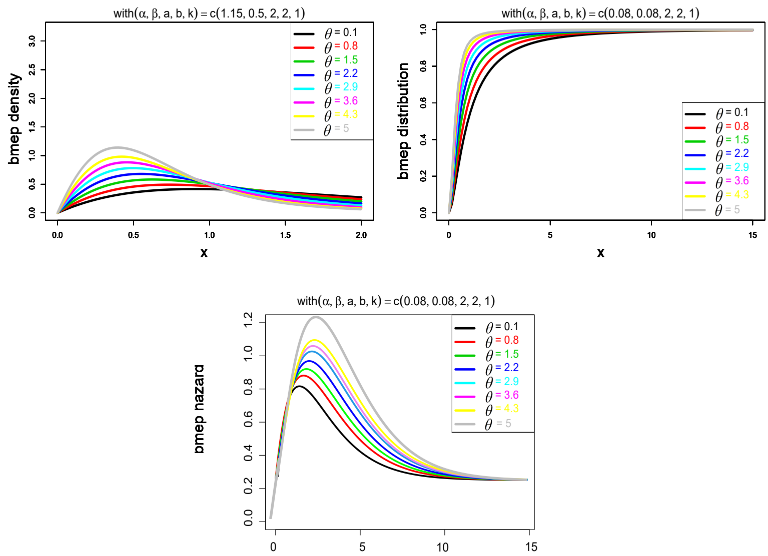

Figure 1: Plots of the density function, cumulative distribution function and survival function of the BMEP distribution for different values of the vector $(\alpha, \beta, \theta, a, b, k)$.

unimodal failure rates which are common in reliability and biological studies. In order to identify the type of failure rate of lifetime data, many approaches have been proposed.

2.1.2 Beta Modified Exponential Geometric Distribution

The beta modified exponential geometric (BMEG) distribution is a special case of the BMEPS distributions with $a_n = 1$ and $C(\theta) = \theta(1-\theta)^{-1}$. Using the CDF (2.1), the CDF of the beta modified exponential geometric (BMEG) distribution is given by

$$F_{BMEG}(x) = 1 - \frac{\Big(\theta\big(I_{F(x)}(\alpha,\beta)\big)^k\Big)\Big(1-\big(\theta\big(I_{F(x)}(\alpha,\beta)\big)^k\big)\Big)^{-1}}{\theta(1-\theta)^{-1}},$$

where $\alpha, \beta > 0$, $0 \leq \theta \leq 1$ and $a, b \geq 0$. The PDF of BMEG $f_{BMEG}(x)$ is

$$\frac{B_{k,\theta}(\frac{1-e^{-\beta x}}{A_{\alpha,\beta}})^{(a-1)}(\frac{\alpha e^{-\beta x}}{A_{\alpha,\beta}})^{(b-1)}(\frac{\alpha\beta e^{-\beta x}}{(A_{\alpha,\beta})^2})(I_{F(x)}(\alpha,\beta))^{k-1}\Big(1-\theta(I_{F(x)}(\alpha,\beta))^k\Big)^{-2}}{\theta(1-\theta)^{-1}}.$$

The plots of the PDF, CDF and hazard rate function of the BMEG distribution for some values of $\alpha, \beta, \theta, a, b$ and k are given in Figure 2. The hazard rate function

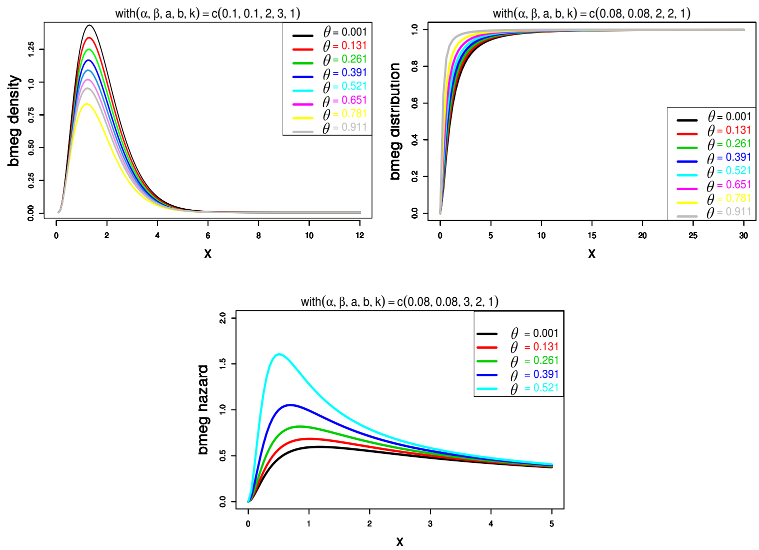

Figure 2: Plots of the density function, cumulative distribution function and hazard rate function of the BMEG distribution for different values of the vector $(\alpha, \beta, \theta, a, b, k)$.

of the BMEG distribution is given by

$$h_{BMEG}(x) = \frac{B_{k,\theta}(\frac{1-e^{-\beta x}}{A_{\alpha,\beta}})^{(a-1)}(\frac{\alpha e^{-\beta x}}{A_{\alpha,\beta}})^{(b-1)}(\frac{\alpha\beta e^{-\beta x}}{(A_{\alpha,\beta})^2})\Big(1-\theta(I_{F(x)}(\alpha,\beta))^k\Big)^{-2}}{\Big(I_{F(x)}(\alpha,\beta)\Big)\Big(1-(\theta\big(I_{F(x)}(\alpha,\beta))^k\big)\Big)^{-1}}.$$

2.1.3 Beta Modified Exponential Logarithmic Distribution

The beta modified exponential logarithmic (BMEL) distribution is a special case of the BMEPS distributions with $a_n = \frac{1}{n}$ and $C(\theta) = -log(1-\theta)$. Using the CDF (2.1), the CDF of the BMEL distribution is given by

$$F_{BMEL}(x) = 1 - \frac{log\Big(1-\theta\big(I_{F(x)}(\alpha,\beta)\big)^k\Big)}{log(1-\theta)} \tag{2.2}$$

where $\alpha, \beta > 0$, $0 \leq \theta \leq 1$ and $a, b, k \geq 0$.
The plots of the PDF, CDF and hazard rate function of the BMEL distribution for some values of $\alpha, \beta, \theta, a, b$ and k are given in Figure 3. The associated PDF of BMEL and hazard rate function are obtained by the equation 2.2, respectively, by

$$f_{BMEL}(x) = \frac{B_{k,\theta}(\frac{1-e^{-\beta x}}{A_{\alpha,\beta}})^{(a-1)}(\frac{\alpha e^{-\beta x}}{A_{\alpha,\beta}})^{(b-1)}(\frac{\alpha\beta e^{-\beta x}}{(A_{\alpha,\beta})^2})(I_{F(x)}(\alpha,\beta))^{k-1}(1-\theta\big(I_{F(x)}(\alpha,\beta)\big)^k)^{-1}}{log(1-\theta)},$$

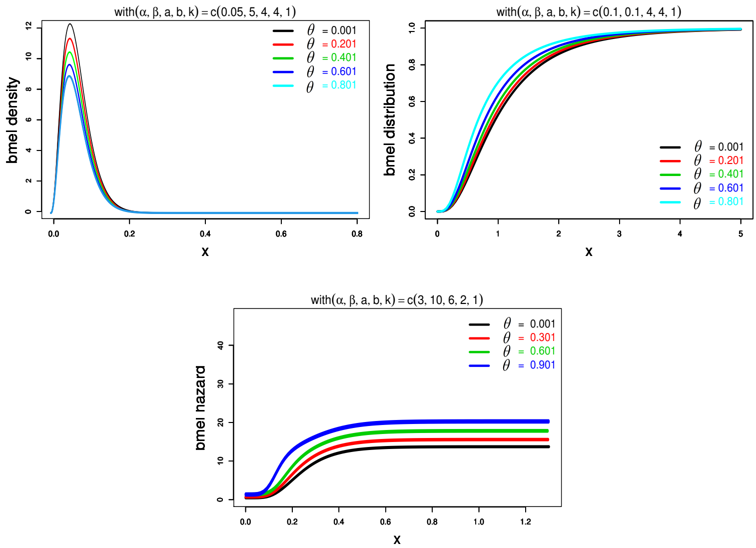

Figure 3: Plots of the density function, cumulative distribution function and hazard rate function of the BMEL distribution for different values of the vector $(\alpha, \beta, \theta, a, b, k)$.

and

$$h_{BMEL}(x) = \frac{B_{k,\theta}(\frac{1-e^{-\beta x}}{A_{\alpha,\beta}})^{(a-1)}(\frac{\alpha e^{-\beta x}}{A_{\alpha,\beta}})^{(b-1)}(\frac{\alpha\beta e^{-\beta x}}{(A_{\alpha,\beta})^2})(I_{F(x)}(\alpha,\beta))^{k-1}(1-\theta\big(I_{F(x)}(\alpha,\beta)\big)^k)^{-1}}{log\Big(1-\theta\big(I_{F(x)}(\alpha,\beta)\big)^k\Big)}.$$

2.1.4 Beta Modified Exponential Binomial Distribution

The beta modified exponential binomial (BMEB) distribution is a special case of BMEPS distributions with $a_n = \binom{m}{n}$, $C(\theta) = (\theta+1)^m - 1$ and $C'(\theta) = m(\theta+1)^{m-1}$. Using CDF (2.1), the CDF of BMEB distribution is given by

$$F_{BMEB}(x) = 1 - \frac{\big(\theta\big(I_{F(x)}(\alpha,\beta)\big)^k + 1\big)^m - 1}{(\theta+1)^m - 1}$$

where $\alpha, \beta > 0$, $0 \leq \theta \leq 1$ and $a, b, k \geq 0$. The PDF of BMEB is

$$f_{BMEB}(x) = \frac{B_{k,\theta}(\frac{1-e^{-\beta x}}{A_{\alpha,\beta}})^{(a-1)}(\frac{\alpha e^{-\beta x}}{A_{\alpha,\beta}})^{(b-1)}(\frac{\alpha\beta e^{-\beta x}}{(A_{\alpha,\beta})^2})(I_{F(x)}(\alpha,\beta))^{k-1}\Big(m(\theta\big(I_{F(x)}(\alpha,\beta)\big)^k+1)^{m-1}\Big)}{(\theta+1)^m - 1}.$$

The hazard rate function of BMEB distribution is given by

$$h_{BMEB}(x) = \frac{B_{k,\theta}(\frac{1-e^{-\beta x}}{A_{\alpha,\beta}})^{(a-1)}(\frac{\alpha e^{-\beta x}}{A_{\alpha,\beta}})^{(b-1)}(\frac{\alpha\beta e^{-\beta x}}{(A_{\alpha,\beta})^2})(I_{F(x)}(\alpha,\beta))^{k-1}\Big(m(\theta(I_{F(x)}(\alpha,\beta))^k+1)^{m-1}\Big)}{((\theta(I_{F(x)}(\alpha,\beta))^k+1)^m-1}.$$

In the following, we derive the quantiles and moments of BMEPS distributions, alongside elucidating the probability density function of the ith order statistic.

3. Statistical properties

In this section, we propose some of the basic statistical properties of the BMEP. For example, we provide quantiles and order statistic, Renyi and Shannon entropies, as well as moments. The moment generating function, residual life function, probability weighted moments, mean, deviations and Bonferroni and Lorenz curves are also provided for the BMEP.

3.1 Quantiles, moments and order statistics

The quantiles of a distribution can be used in data generation from this distribution. The quantile x_q of the BMEPS$(\alpha, \beta, a, b, k, \theta)$ is the real solution of the following equation:

$$\frac{B_{k,\theta}(\frac{1-e^{-\beta x_q}}{A_{\alpha,\beta,q}})^{(a-1)}(\frac{\alpha e^{-\beta x_q}}{A_{\alpha,\beta,q}})^{(b-1)}(\frac{\alpha\beta e^{-\beta x_q}}{(A_{\alpha,\beta,q})^2})(I_{F(x_q)}(\alpha,\beta))^{k-1}C'\Big(\theta\big(I_{F(x_q)}(\alpha,\beta)\big)^k\Big)}{C(\theta)} = 0$$

The above equation has no closed form solution in x_q, so we have to use a numerical technique such as the Newton-Raphson method to get the quantile.
The PDF $f_{i:n}$ of the ith order statistic for a random sample $X_1, X_2, ..., X_n$ from the BMEPS distribution is given by

$$\begin{aligned} f_{i:n}(x) &= \frac{1}{B(i,n-i+1)} f(x)F(x)^{i-1}[1-F(x)]^{n-i} \\ &= \frac{1}{B(i,n-i+1)} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \left[\frac{C(\theta\big(I_{F(x)}(\alpha,\beta)\big)^k)}{C(\theta)}\right]^{j+i-1} \end{aligned}$$

and the CDF is

$$\begin{aligned} F_{i:n}(x) &= \sum_{k=i}^{n} \binom{n}{k} F(x)^k [1-F(x)]^{n-k} \\ &= \sum_{k=i}^{n} \sum_{j=0}^{n-k} \binom{n-k}{j} \binom{n}{k} (-1)^j \left[\frac{C(\theta\big(I_{F(x)}(\alpha,\beta)\big)^k)}{C(\theta)}\right]^{j+k}. \end{aligned}$$

Carrasco et al. (2008) obtained an infinite representation for the rth moment of the $BME(\alpha, \beta, a, b, k)$ distribution. If Y has the $BME(\alpha, \beta, a, b, k)$, the rth moment of Y say ν_r, is given as follows

$$\nu_r[\alpha,\beta,a,b,k] = \sum_{j=0}^{\infty}\sum_{i_1,\dots,i_r=1}^{\infty} w_j \frac{A_{i_1,\dots,i_r}\Gamma(\frac{S_r}{2})}{(\alpha(j+b))^{S_r}} \tag{3.3}$$

where $A_{i_1,\dots,i_r}=a_{i_1}\dots a_{i_r}$, $S_r=a_{i_1}+\dots+a_{i_r}$,

$$w_j=\frac{(-1)^j\Gamma(a)}{B(a,b)\Gamma(a-j)(b+j)j!}$$

and

$$a_i=\frac{(-1)^{i+1}i^{i-2}}{(i-1)!}.$$

Let $Y_{i:n}$ be the ith order statistic of a random sample from the BME distribution. The rth moment of BMEPS$(\alpha,\beta,a,b,k,\theta)$, is given as follows

$$\begin{aligned}\mu_r=E(X^r) &= \sum_{n=1}^{\infty}P(N=n)E(Y_{(n)}^r)\\ &= \sum_{n=1}^{\infty}\frac{a_n\theta^n}{C(\theta)}n\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(a)}{B(a,b)\Gamma(a-j)(b+j)j!}\sum_{i_1,\dots,i_r=1}^{\infty}\frac{A_{i_1,\dots,i_r}\Gamma(\frac{S_r}{2})}{(\alpha(j+b))^{S_r}}\end{aligned}$$

Based on the results given in (3.3), the measures of skewness and kurtosis of the BMEP(α,a,b,k,θ) can be obtained according to the following relations, respectively,

$$Skewness=\frac{\mu_3-3\mu_1\mu_2+2{\mu_1}^3}{(\mu_2-\mu_1)^{\frac{3}{2}}},$$

$$Kurtosis=\frac{\mu_4-4\mu_3\mu_1+6{\mu_1}^2\mu_2-3{\mu_1}^4}{(\mu_2-\mu_1)^2}.$$

To illustrate the behavior of the skewness and kurtosis when θ varies, Figure 4 presents the Galton's skewness Johnson et al. (1995) and Moors' kurtosis Moors (1988) as functions of θ for selected values of α, β,a,b and k.

3.2 Mean deviations

The amount of scatter in a population can be measured by the totality of deviations from the mean and median. The mean deviation from the mean is a robust statistic, being more resilient to outliers in a data set than the standard deviation. For a random variable X with PDF $f(x)$, CDF $F(x)$, mean μ and median M, the deviation from the mean and the mean deviation from the median are defined by

$$\delta_1(x)=\int_0^{\infty}\mid x-\mu\mid f(x)dx=2\mu F(\mu)-2I(\mu)$$

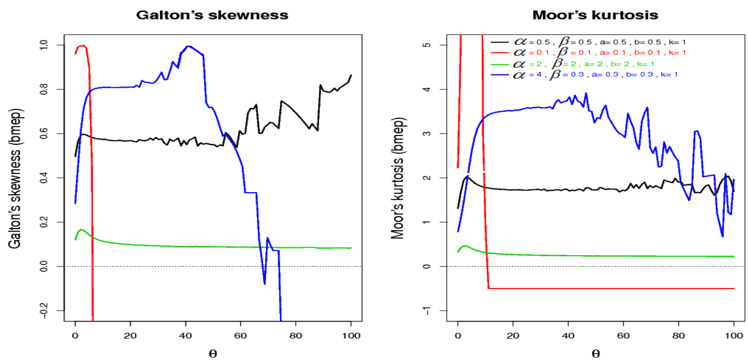

Figure 4: The effect of θ on the Galton's skewness and Moors' kurtosis for different values of α, β, a, b and k.

and

$$\delta_2(x) = \int_0^{\infty} \mid x - M \mid f(x)dx = 2MF(M) - M + \mu - 2I(M)$$

respectively, where

$$I(a) = \int_0^{a^*} xf(x)dx.$$

4. Estimation and inference

Standard statistical techniques such as the method of maximum likelihood can always be used for parametric estimation. The likelihood equations, given the complete or censored failure data set, can be derived and solved.

4.1 The Maximum Likelihood Estimators

Parameter estimation is usually a difficult problem, specially for a six parameter BMEPS distribution. Methods like the maximum likelihood estimation will not yield a closed form solution. Different methods can be used to estimate the model parameters. Among these methods, the Maximum Likelihood Estimation method is the most commonly used method for model estimation. In this subsection, we use the maximum likelihood procedure to derive the point and interval estimates of the parameters.

$$\begin{aligned}
L &= n\ln(k)+n\ln(\theta)+n\ln(\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)})+(a-1)\sum_{i=1}^{n}\ln(\frac{1-e^{-\beta x_i}}{1-(1-\alpha)e^{-\beta x_i}}) \\
&+(b-1)\sum_{i=1}^{n}\ln(\frac{\alpha e^{-\beta x_i}}{1-(1-\alpha)e^{-\beta x_i}})+\sum_{i=1}^{n}\ln(\frac{\alpha\beta e^{-\beta x_i}}{(1-(1-\alpha)e^{-\beta x_i})^2}) \\
&+(k-1)\ln(I_{F(x)}(\alpha,\beta))+n\ln\left(C^{'}(\theta(I_{F(x)}(\alpha,\beta))^k)\right)-\ln(C(\theta))
\end{aligned}$$

Calculating the first partial derivatives of L with respect to $\alpha,\beta,a,b,k,\theta$ and equating each to zero.
To find out the maximum likelihood estimators of $\alpha,\beta,a,b,k,\theta$, we have to solve the system of nonlinear equations with respect to α,β,a,b,k and θ. As it seems, this system has no closed form solution in $\alpha,\beta,a,b,k,\theta$. Then, we have to use a numerical technique method, such as the Newton-Raphson method to obtain the solution.

4.2 Asymptotic Confidence Bounds

The approximate confidence intervals of the parameters based on the asymptotic distributions of the MLE of the parameters $\alpha,\beta,a,b,k,\theta$ are derived in this subsection. Then the observed information matrix is obtained as

$$\mathbf{I}=-\begin{pmatrix}
\frac{\partial^2 L}{\partial\alpha^2} & \frac{\partial^2 L}{\partial\alpha\partial\beta} & \frac{\partial^2 L}{\partial\alpha\partial a} & \frac{\partial^2 L}{\partial\alpha\partial b} & \frac{\partial^2 L}{\partial\alpha\partial k} & \frac{\partial^2 L}{\partial\alpha\partial\theta} \\
\frac{\partial^2 L}{\partial\beta\partial\alpha} & \frac{\partial^2 L}{\partial\beta^2} & \frac{\partial^2 L}{\partial\beta\partial a} & \frac{\partial^2 L}{\partial\beta\partial b} & \frac{\partial^2 L}{\partial\beta\partial k} & \frac{\partial^2 L}{\partial\beta\partial\theta} \\
\frac{\partial^2 L}{\partial a\partial\alpha} & \frac{\partial^2 L}{\partial a\partial\beta} & \frac{\partial^2 L}{\partial a^2} & \frac{\partial^2 L}{\partial a\partial b} & \frac{\partial^2 L}{\partial a\partial k} & \frac{\partial^2 L}{\partial a\partial\theta} \\
\frac{\partial^2 L}{\partial b\partial\alpha} & \frac{\partial^2 L}{\partial b\partial\beta} & \frac{\partial^2 L}{\partial b\partial a} & \frac{\partial^2 L}{\partial b^2} & \frac{\partial^2 L}{\partial b\partial k} & \frac{\partial^2 L}{\partial b\partial\theta} \\
\frac{\partial^2 L}{\partial k\partial\alpha} & \frac{\partial^2 L}{\partial k\partial\beta} & \frac{\partial^2 L}{\partial k\partial a} & \frac{\partial^2 L}{\partial k\partial b} & \frac{\partial^2 L}{\partial k^2} & \frac{\partial^2 L}{\partial k\partial\theta} \\
\frac{\partial^2 L}{\partial\theta\partial\alpha} & \frac{\partial^2 L}{\partial\theta\partial\beta} & \frac{\partial^2 L}{\partial\theta\partial a} & \frac{\partial^2 L}{\partial\theta\partial b} & \frac{\partial^2 L}{\partial\theta\partial k} & \frac{\partial^2 L}{\partial\theta^2}
\end{pmatrix},$$

The variance-covariance matrix may be approximated as $\Sigma = I^{-1}$. Since Σ involves the parameters α,β,a,b,k and θ, we replace the parameters by the corresponding MLE's in order to obtain an estimate of Σ, which is denoted by $\widehat{\Sigma}=\widehat{I}^{-1}$, where $\widehat{I}_{ij}=I_{ij}$ with $(\hat{\alpha},\hat{\beta},\hat{a},\hat{b},\hat{k},\hat{\theta})$ substituting $(\alpha,\beta,a,b,k,\theta)$. By using this approximation, approximate $100(1-\delta)\%$ confidence intervals for $\alpha,\beta,a,b,k,\theta$ are determined, respectively, as

$$\widehat{\alpha}\pm z_{\frac{\delta}{2}}\sqrt{\widehat{\Sigma}_{11}}\quad,\quad\widehat{\beta}\pm z_{\frac{\delta}{2}}\sqrt{\widehat{\Sigma}_{22}}\quad,\quad\widehat{a}\pm z_{\frac{\delta}{2}}\sqrt{\widehat{\Sigma}_{33}}\quad,\quad\widehat{b}\pm z_{\frac{\delta}{2}}\sqrt{\widehat{\Sigma}_{44}},$$

$$\widehat{k}\pm z_{\frac{\delta}{2}}\sqrt{\widehat{\Sigma}_{55}}\quad and\quad\widehat{\theta}\pm z_{\frac{\delta}{2}}\sqrt{\widehat{\Sigma}_{66}}$$

where z_δ is the upper $\delta{-}th$ percentile of the standard normal distribution. We shall now move to hypothesis testing inference on the parameters that index the BMEPS law. Let $\Delta=[\alpha,\beta,a,b,k,\theta]$, $\Delta_1=[\alpha,\beta,a,b,k]$ and $\Delta_2=[\theta]$ so $\Delta=[\Delta_1,\Delta_2]$.

Suppose we wish to test the hypothesis $H_0 : \Delta_2 = \Delta_{2\circ}$, against the alternative hypothesis $H_1 : \Delta_2 \neq \Delta_{2\circ}$. To that end, we can use the likelihood ratio (LR) test whose test statistic is given by $G = 2[\ell(\widehat{\Delta}) - \ell(\widetilde{\Delta})]$, where $\widehat{\Delta} = [\widehat{\Delta_1}, \widehat{\Delta_2}]$ and $\widetilde{\Delta} = [\widetilde{\Delta_1}, \Delta_{2\circ}]$ denote the MLEs of Δ under the null and the alternative hypotheses, respectively. Under the null hypothesis, G is asymptotically (as $n \to \infty$) distributed as ${\chi_k}^2$, where k is the dimension of the vector Δ_2 of parameters of interest. We reject the null hypothesis at the nominal level δ $(0 < \delta < 1)$ if $G > {\chi^2}_{k,1-\delta}$, where ${\chi^2}_{k,1-\delta}$ is the $1 - \delta$ quantile of ${\chi^2}_k$. Using this test, one can select between a BMEPS and an BME model, which can be done by testing $H_0 : \Delta_2 \downarrow_0$.

5. Application

From the current study, it is hoped that the BMEPS distribution can be used more widely in both theoretical and applied contexts. In this section, we analyze a real data set to demonstrate the performance of the BMEPS distribution in practice. The data set is a sample of 50 components taken from Aarset (1987). These data were also analyzed in Choulakian and Stephens (2001).

We illustrate the superiority of the new distribution compared to some of its sub-models. We then perform the goodness of fit analysis of the BMEL distribution and sub-models, which allows their evaluation relative to each other and to the more general BMEL model. However, the lower values of AIC and BIC for the BMEL and other distributions indicate that these models might be chosen as the best fits for the data. In addition to comparing the models, we use two other criteria. First, we consider the LR statistic, and next, we consider formal goodness-of-fit tests. The required numerical evaluations are implemented using R software.

We consider the widely used data from Aarset (1987), also reported in Mudholkar and Srivastava (1993), and Wang (2000), on lifetimes of 50 components that possess a bathtub-shaped failure rate property. The data contain the times to failure of 50 devices put on a life test at time 0, from Aarset (1987) as in Table 2. Also, Table 3 shows descriptive statistics of the Aarset data, and Figure 5 shows the

Table 2: Lifetimes of 50 devices

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

Table 3: Descriptive values of Aarset data

Min	1st.Qu	Median	Mean	3rd.Qu	Max	Skewness	Kurtosis
0.10	13.50	48.50	45.69	81.25	86.00	-0.14	1.41

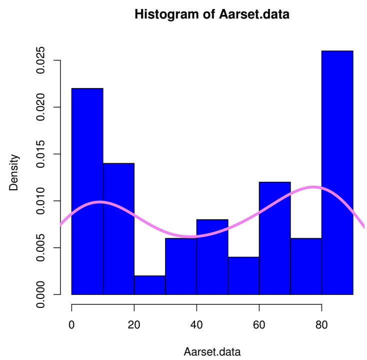

Figure 5: Plots of density function, cumulative distribution function and hazard function.

histogram and the approximation of the density curve of the data.

Table 4 presents the maximum likelihood estimates (MLEs) of the parameters for each model, accompanied by the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and the value of $-2\log(L)$. These statistical measures are crucial for model comparison, where lower values typically indicate a better fit to the data. Among the fitted models, the BMEL distribution stands out with the lowest AIC, BIC, and $-2\log(L)$ values (509.080, 520.560, and 497.080, respectively). These results suggest that the BMEL distribution provides the most parsimonious fit to the data, balancing model complexity with goodness of fit. Notably, the BMEG distribution, with slightly higher AIC and BIC values (494.500 and 505.980), also performs well, indicating that it captures the underlying data structure effectively.

The substantial differences in AIC and BIC values between the best models and others, such as the BMEP distribution (AIC = 5709.970, BIC = 5721.440), further reinforce the BMEL distribution's superiority in this context. The use of these criteria allows for a more nuanced model selection process, where both the goodness of fit and the number of estimated parameters are considered. Moreover, the consistency between AIC and BIC rankings supports the robustness of the BMEL distribution as the best model for the Aarset data. However, it is important to

Table 4: MLEs of the model parameters, and the measures of AIC ,BIC and $-2log(L)$ for Aarset data (values of best fitted model are highlighted).

	Estimates						Statistic		
Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{k}$	$\hat{\theta}$	AIC	BIC	$-2log(L)$
BMEP	0.016	0.596	7.000	1.999	6.000	0.000	5709.970	5721.440	5697.970
BMEG	1.005	0.004	0.127	3.299	5.398	0.000	494.500	505.980	482.500
BMEL	1.948	0.238	2.868	0.055	0.365	0.000	509.080	520.560	497.080
MEP	3.693	0.030			0.583	0.000	477.260	484.910	469.260
MEG	29.492	0.050			0.421	0.00	466.420	474.070	458.420
MEL	4.027	0.0310			0.598	0.000	476.530	484.180	468.530
EP	0.015				0.000	0.551	489.710	495.440	483.710
EG	0.015				0.000	0.565	489.220	494.960	483.220
EL	0.017				0.000	0.688	486.500	492.230	480.500
BME	0.001	3.225	0.500	0.007	1.701	2.621	478.200	487.760	468.200
ME	0.033				0.022		483.110	486.930	479.110
E				0.016	0.596	7.000	484.180	486.090	482.180

note that while the BMEL model outperforms others based on these criteria, the BMEG and MEP distributions also demonstrate strong potential, depending on the specific requirements of the analysis or application. The results underline the importance of considering multiple models and criteria when performing survival analysis, particularly in contexts where the underlying data may exhibit complex behavior, such as varying failure rates or multimodal distributions. In general, the statistical evidence strongly favors the BMEL distribution, but the performance of models like BMEG and MEP suggests they may also be viable in similar contexts. On the other hand, we can compute the maximum values of the unrestricted and restricted log-likelihoods to derive the LR statistics, allowing us to test sub-models of the BMEPS distribution. For instance, the test between the BME distribution and the BMEL model examines $H_0 : \theta \downarrow_0$ versus $H_1 : \theta \not\downarrow_0$ $(\theta > 0)$. Typically, we define $\Theta = (\Theta_1, \Theta_2)$, partitioning the BMEPS distribution's parameter space into subsets where Θ_1 denotes parameters of interest and Θ_2 signifies the remaining parameters. Using $\widehat{\Theta}$ and $\widetilde{\Theta}$ as the maximum likelihood estimates (MLEs) under alternative and null hypotheses respectively, with Θ_1^0 as a specified parameter vector, we compute the LR statistic for testing the null hypothesis $H_0 : \Theta_1 = \Theta_1^0$ against the alternative $H_1 : \Theta_1 \neq \Theta_1^0$.
Consequently, under the null hypothesis of $\theta \downarrow_0$, the computed test statistic is LR-statistics $= -28.883$. Given that the p-value (1.000) exceeds 1%, we confidently fail to reject our null hypothesis, implying that the parameter $\theta \downarrow_0$. This suggests that the BME distribution adequately models the device failure data. Table 5 presents the values of several more LR statistics for reference.

Table 5: LR tests for Aarset data.

Model	LR-statistics	$p - value$	Model	LR statistics	$p - value$
BMEP versus MEP	-528.705	1.000	BMEG versus ME	-3.396	1.000
BMEP versus EP	-524.264	1.000	BMEG versus E	-0.325	0.71
BMEP versus BME	-529.769	1.000	BMEL versus MEL	-28.555	1.000
BMEP versus ME	-5218.862	1.000	BMEL versus EL	-16.585	1.000
BMEP versus E	-5215.790	1.000	BMEL versus BME	-28.883	1.000
BMEG versus MEG	-24.083	1.000	BMEL versus ME	-17.975	1.000
BMEG versus EG	0.720	0.869	BMEL versus E	-14.904	1.000
BMEG versus BME	-14.303	1.000			

Table 6: Goodness-of-fit tests for Aarset data.

	Statistic					Statistic			
Distribution	AD^*	W^*	K-S	$p - value$	Distribution	AD^*	W^*	$K - S$	$p - value$
BMEP	7.341	14.391	0.901	0.000	EP	4.143	0.802	0.235	0.008
BMEG	3.391	0.623	0.212	0.022	EG	3.981	0.767	0.231	0.009
BMEL	5.478	0.967	0.261	0.002	EL	3.407	0.625	0.214	0.021
MEP	2.539	0.437	0.182	0.073	BME	101.597	7.498	0.760	0.000
MEG	1.599	0.238	0.140	0.28	ME	4.117	0.437	0.163	0.140
MEL	2.459	0.411	0.177	0.085	E	3.708	0.524	0.191	0.052

Next, we examine the Cramer-von Mises (W^*) and Anderson-Darling (A^*) statistics, where smaller values of these statistics generally indicate a better fit to the data. Detailed descriptions of the statistics W^* and A^* can be found in Chen and Balakrishnan (1995). Table 6 provides the goodness-of-fit test results for the Aarset data, including the Anderson-Darling (A^*) and Cramer-von Mises (W^*) statistics, along with the Kolmogorov-Smirnov ($K - S$) test statistic and corresponding p-values for each model. These statistics are crucial in assessing how well the fitted distributions capture the underlying data characteristics, where smaller values generally suggest a better fit. Among the models considered, the MEG distribution exhibits the lowest A^* and W^* statistics (1.599 and 0.238, respectively), indicating a superior fit to the data compared to other distributions. The K-S statistic for the MEG model (0.140) further supports its adequacy, with a high p-value of 0.28, suggesting no significant deviation from the observed data.

In contrast, the BMEP distribution, despite its flexibility, shows much higher A^* and W^* values (7.341 and 14.391, respectively), indicating a poorer fit. The corresponding $K - S$ statistic (0.901) and p-value (0.000) reinforce this, signaling a significant departure from the observed data distribution. The BMEL distribution, while not performing as well as the MEG model, still provides a reasonable fit with A^* and W^* values of 5.478 and 0.967, respectively. Its K-S statistic (0.261) and p-value (0.002) indicate some level of deviation, but it is less pronounced than

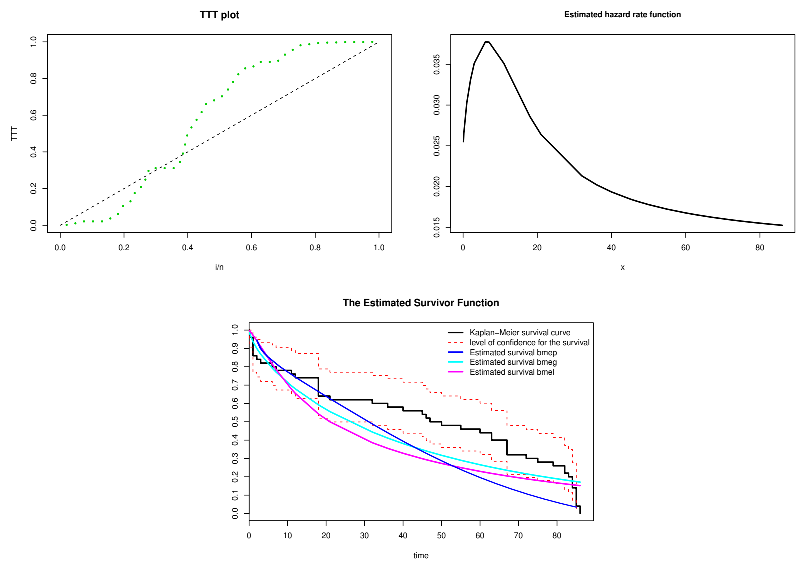

Figure 6: Left panels: empirical TTT-plot. Right panels: Estimated hazard rate function. Bottom panels: Estimated survival functions for three fitted models and the empirical survival function. All for Aarset data.

in the BMEP model.

Also, the MEG distribution emerges as the most suitable model based on these goodness-of-fit statistics, providing the best balance between simplicity and accuracy in capturing the underlying data structure. This is particularly evident when comparing the MEG distribution to the more complex BMEP and BMEL distributions, which, despite their theoretical flexibility, do not perform as well in practice. The results underline the importance of selecting a model that not only theoretically accommodates diverse failure rate patterns but also empirically fits the data effectively.

Overall, while models like BMEL offer theoretical advantages, the empirical evidence strongly favors the MEG distribution for the Aarset data. This emphasizes the need for a careful balance between model complexity and goodness-of-fit when selecting the most appropriate distribution for survival analysis.

Figure 6 presents a comprehensive visual analysis of the Aarset data, featuring the empirical Total Time on Test (TTT) plot, estimated hazard rate functions, survival functions, and the Kaplan–Meier curve for the three fitted models. In the left panels, the empirical TTT-plot offers insight into the underlying failure rate pattern of the data. The estimated hazard rate functions, shown in the right panels, reveal the behavior of each model in capturing the risk over time. Notably,

the BMEG distribution aligns closely with the empirical hazard rate, indicating its capability to model the failure process effectively.
The bottom panels display the estimated survival functions alongside the empirical survival function derived from the Kaplan–Meier estimator. Here, the BMEG distribution again demonstrates its superiority, as its estimated survival function closely follows the empirical curve, suggesting an accurate representation of the data's survival characteristics. Overall, the graphical evidence reinforces the earlier statistical findings, highlighting the BMEG distribution as the most appropriate model for the Aarset data. Its strong performance across the empirical TTT-plot, hazard rate estimation, and survival function fitting underscores its practical utility in survival analysis, making it a robust choice for modeling lifetime data in this context.

Discussion and Resuls

We introduced the Beta Modified Exponential Power Series (BMEPS) Distribution, a six-parameter lifetime model that amalgamates features from the beta modified Exponential and Power Series distributions. This BMEPS model, explored in our study, demonstrated remarkable adaptability, accommodating various types of failure data, including the potential for a bathtub-shaped failure rate function. Our thorough mathematical treatment covered order statistics, providing explicit expressions for the density function and moments. We examined diverse properties of the BMEPS distribution, investigated quantiles and moments, and employed the EM-algorithm for maximum likelihood estimation. By fitting the BMEPS model to real-world data, we demonstrated its practical application. Envisioning broader utility in survival analysis, life distributions, and among reliability engineers, we anticipated widespread interest and adoption of our extended model.

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