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Research paper

Asset-liability management for with-profit life insurance policies: A novel multi-stage stochastic programming model

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Abstract:
Asset-liability management (ALM) is a critical issue for insurance companies because the premiums received from policyholders should be invested according to regulatory frameworks while providing suitable profitability, and simultaneously, the insurer should fulfill its obligations to policyholders on time. Our focus is on participating (with-profit) life insurance policies, where policyholders not only receive a guaranteed profit but also participate in the return of the insurer's investment-portfolio. Due to the risks of death and surrender, uncertainty in asset returns, the broad range of insurance products and regulations, it is difficult to make optimal decisions. In this paper, we aim to present a new multi-stage stochastic programming ALM model for with-profit life insurance policies. Compared to existing models that involve some simplifications, our model incorporates more details and is closer to reality. Specifically, our model is multi-stage and updates the amount of policies investment reserves based on the realized return of the investment-portfolio. Evaluation of the model across a variety of datasets confirms the effectiveness of the proposed model.

Keywords: Asset-liability management; participating (with-profit) life insurance policies; Multi-stage stochastic programming; Scenario; Value-of-stochastic-solution.
JEL Classifications: 90C15, 90C90.

1 Introduction

Asset-liability management (ALM) is an important problem for insurance companies. On the one hand, the premiums received from policyholders should be invested to provide suitable profit while satisfying regulatory frameworks. On the other hand, the insurer must fulfill its obligations to policyholders. In this paper, we concentrate on participating (with-profit) life insurance policies, which constitute a significant portion of the life insurance market. These policies not only provide policyholders with a guaranteed profit but also allow them to share in the return

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of the insurer's investment-portfolio. Insurance companies face the risks of death and surrender, as well as the uncertainty in asset returns and hence, due to the wide range of insurance products and regulations, making appropriate decisions is complex. Optimization- and simulation-based models play a crucial role in this context and have attracted significant attention from researchers. In the following, first, the studies addressing ALM in the context of with-profit life insurance using optimization and simulation approaches, are reviewed. Then, the contributions of this paper are discussed.

1.1 Literature review

Gerstner, et al. [9], Fernández, et al. [7], and Orreborn [15] focused on simulation-based models to show the evolution of asset- and liability-portfolios. Huang and Lee [12] addressed the asset-portfolio optimization for an insurer providing life insurance policies in single-period and multi-period cases focusing on the first two momentums of the accumulated assets value. Gurin [11] studied the asset-portfolio management in companies providing life insurance policies and evaluated the impact of considering credit risk and interest-rate risk in comparison to the variance risk measure used in the well-known mean-variance model of Markowitz [13]. Bohnert, et al. [2] investigated the impact of asset and liability composition on different risk measures. Gülpnar, et al. [10] utilized robust optimization approach for ALM. Chen, et al. [3] proposed a bi-level programming model under uncertainty in which in addition to asset-portfolio optimization, the policy product allocation is also decided.

Some studies addressed replicating portfolio (RP) for asset management in life insurance companies. An RP is a collection of financial instruments designed to approximately replicate the present values of liabilities based on a large number of economic scenarios. Insurance companies can utilize RPs to align assets and liabilities and consequently, manage risks. See amongst others Dauland and Vidal [4], Adelmann, et al. [1], Devineau and Chauvigny [5], and Natolski and Werner [14].

ALM is a multi-period problem affected by uncertainty, and an efficient technique for its modeling, is the multi-stage stochastic programming (MSP), known as a successful approach for optimal decision-making under uncertainty. Francesco and Simonella [8], Rao, et al. [16], and Dutta, et al. [6] presented an MSP model for asset-liability management in life insurance. Specifically, Francesco and Simonella [8] proposed a hybrid approach based on optimization and Monte-Carlo simulation, assuming that the liability-portfolio consists of with-profit life insurance policies, evolving based on policyholder savings accounts and surrender and biometric mortality models. The asset-portfolio consists of participation bonds, stocks, and cash, and the evolution of assets and liabilities is simulated using appropriate stochastic models. The policyholder's savings account benefits from either the guaranteed return or the participatory return, whichever is higher. The cash flow corresponding to new policyholders is also considered. The asset-portfolio is adjusted by solving a constrained nonlinear optimization model in which the gap between asset and liability durations is minimized subject to meeting a threshold return and budget restriction. Rao, et al. [16] presented MSP model and a decision-support system with the aim of maximizing the total expected reserve at the end of the planning horizon. A similar problem was addressed by Dutta, et al. [6] while analyzing the required number of scenarios and its effect on the solution stability.

1.2 Main contributions

Given the performance of MSP technique in modeling an uncertain multi-period problem, this paper aims to present a new MSP model for the ALM problem in insurance companies providing with-profit insurance policies. As mentioned earlier, the optimization model of Francesco and Simonella [8], determines portfolio adjustment decisions by considering merely the events of a single future period; in contrast, our model has a multi-stage nature and considers a multi-period decision-making horizon which can lead to well utilization of investment opportunities and better management in fulfilling obligations. Additionally, while Rao, et al [16] and Dutta, et al. [6] treated the investment reserve of each policy as a parameter, our model recognizes that the participatory return in each period depends on the performance of the investment-portfolio during that period. Additionally, the investment reserve of policyholders in each period is influenced by the participatory return of that period. Hence, both quantities (the return of the investment-portfolio and the policyholders' investment return) should be treated as variables.

In this paper, we aim to overcome the shortcomings of existing models by presenting a new MSP model. The main contributions of this paper are as follows: first, a new MSP model is presented for the ALM problem in an insurer providing with-profit life policies. We use a discrete-time model in which liabilities and assets are periodically updated. It is a nonlinear programming (NLP) that, compared to existing models which involves some simplifications, includes more details and is closer to reality. Evaluating the model on a variety of datasets confirms its significance.

The remainder of this section is organized as follows: Section 2 describes the problem in more details. Our novel MSP model is presented in Section 3. Computational experiments are reported in Section 4. Finally, Section 5 concludes and offers directions for future research.

2 Problem description

A with-profit life insurance policy is a type of life insurance contract that not only provides a death benefit to the beneficiaries upon the insured persons death, but also provides an investment opportunity. Policyholders pay premiums regularly, and these amounts (after deducting the costs and fees) are invested by the insurer.

This allows the policyholders to benefit from both guaranteed and participatory investment returns. The returns are added to the policies investment reserves and reinvested in subsequent years, earning compound return. For every with-profit life insurance policy, the start date, maturity date, death benefit, and premium amount are known. Policyholders can receive their investment reserve at the end of the contract (maturity date) or even during the intermediate years (if they choose to surrender the policy). If the insured person dies during the contract period, the death benefit, along with the investment reserve, is paid to the beneficiaries.

When the insurer receives the premium of a with-profit life insurance policy, it first deducts administrative costs. The remaining amount, known as the net premium, is divided into two parts: the death coverage premium and the investment premium. The death coverage premium is set so that the insurer can meet death benefit obligations. In practice, the death coverage premium is calculated based on the insured persons age and health status, according to life tables and the technical interest rate. This involves the calculation of the present value of death benefits paid by the insurer during the policy periods and the present value of the premiums paid by the policyholder during the policy periods. By equating these two amounts, the death coverage premium is determined, and the excess is considered as the investment premium.

The technical interest rate, also called the guaranteed return rate, is set by the government or the regulatory authorities. For example, according to the regulations of the central insurance of Iran (Regulation No. 107 of the Supreme Insurance Council), the technical interest rate is specified as 16% for the first two years of the policy contract, 13% for the next two years, and 10% for the fifth year onwards. Actuaries use this rate to calculate premiums. At the end of each year, companies should calculate the realized return rate. If the realized return rate exceeds the technical interest rate, a portion α of the excess is credited to the policys reserve as the participation return in addition to the guaranteed return. Conversely, if the realized rate is lower than the guaranteed return rate, the policys reserve is credited solely with the technical interest rate.

With respect to the above explanations, consider an insurer offering with-profit life insurance policies. These policies are categorized into several groups, each characterized by similar information including insured ages, policy start and maturity dates, premium amounts, and death benefits. The collection of these policies forms the liability-portfolio, while the asset-portfolio comprises a variety of assets such as bank deposits, units of fixed-income funds, stocks, and gold.

When the policyholder terminates the contract before the maturity date, it is commonly referred to as a surrender request. In this case, the insurer must pay a proportion α' of the accumulated investment reserve up to that moment to the policyholder. Therefore, in addition to the death risk, the insurer also faces surrender risk and hence, it intends to manage its asset-portfolio over the following T periods efficiently. The insurer needs to decide how to adjust the asset-portfolio in each subsequent period to ensure that liability obligations are met while satisfying regulatory requirements. Regulations concerning the asset-portfolio include some upper and lower limits on the percentage of the portfolios value allocated to a specific asset or a group of assets.

Let $\mathbb{T} = \{1, \ldots, T\}$, indexed by t, represent the set of time periods associated with the planning horizon. We use indices $t = 0, -1, -2, \ldots$ to refer to periods before the planning horizon, and $t = T+1, T+2, \ldots$ for periods after that. Assume that $\mathbb{G}$, indexed by g, shows the set of insurance policy groups, where policies within each group $g \in \mathbb{G}$ are similar in terms of the insured person's age, death coverage premium (π_g), investment premium (π'_g), death benefit (d_g), policy start period (τ_g), and maturity period (τ'_g). Further, let $r'_{g,t}$ be the technical interest rate in period t for each insurance policy of group g regarding the regulations.

Assuming that the number of in-flow policies in group g at the beginning of period $t = 1$ equals $n_{g,1}$, the number of insured persons in group g who will die during period 1 ($n'_{g,1}$) can be determined using the life table, given that the insured persons of each group are of the same age. Additionally, assuming that $n''_{g,1}$ is an estimation of the number of surrender requests during period 1, the number of in-flow policies at the beginning of period 2 can be calculated as $n_{g,1} - n'_{g,1} - n''_{g,1}$. Therefore, for each period t, the number of in-flow policies in group g who pay premiums at the beginning of period t is denoted by $n_{g,t}$. From this number, $n'_{g,t}$ persons die and $n''_{g,t}$ individuals surrender during period t. Thus, we have:

$$n_{g,t+1} = n_{g,t} - n'_{g,t} - n''_{g,t} \tag{1}$$

The total death benefit coverage reserve corresponding to all policies in group g that the insurer must have at the beginning of the planning horizon and at the end of period t, are denoted by $e_{g,0}$ and $e_{g,t}$, respectively. The total death benefit coverage reserve for all policies in group g in each period t is equal to the death benefit coverage reserve at the end of period $t-1$, plus its guaranteed technical profit, plus the premiums received from in-flow policies during period t, minus the death benefits paid out during period t:

$$e_{g,t} = \begin{cases} n_g \pi_g - n'_{g,t} d_g & \text{if } t = \tau_g \\ (1 + r'_{g,t}) e_{g,t-1} + n_g \pi_g - n'_{g,t} d_g & \text{if } \tau_g + 1 \leq t \leq \tau'_g \end{cases} \tag{2}$$

Additionally, the minimum investment reserve that each policy of group g must have at the end of period t is equal to the total investment premiums received up to that period plus the guaranteed technical interest earned over the previous periods. This quantity is denoted by $e'_{g,t}$ and is calculated as follows:

$$e'_{g,t} = \begin{cases} \pi'_g & \text{if } t = \tau_g \\ (1 + r'_{g,t}) e'_{g,t-1} + \pi'_g & \text{if } \tau_g + 1 \leq t \leq \tau'_g \end{cases} \tag{3}$$

We denote the investment reserve of group g at the beginning of the planning horizon (calculated based on the maximum of the guaranteed return rate and the realized return rate earned in previous periods) by f_g. We denote the proportion of the investment reserve paid to the policyholder in the case of surrender by α'.

Let $\mathbb{J}$ (indexed by j) be the set of assets (such as bank deposits, units of fixed-income funds, stocks, and gold). Assume that c_j denotes the capital allocated to asset j at the beginning of the planning horizon. According to the regulations, lower and upper limits L_j and U_j are specified on the proportion of the portfolio value allocated to asset j. Additionally, lower and upper limits are defined on the proportion of the portfolio value allocated to specific subsets of assets. For example, according to the regulations of the central insurance of Iran, the total amount invested in a subset containing the bank deposit and the units of fixed-income funds must be at least 15% and at most 60% of the portfolio value. Therefore, assuming that $\mathbb{J}'_o$ (with $o \in \mathbb{O}$) represents these subsets, the lower and upper limits on the proportion of the portfolio value allocated to assets of group $\mathbb{J}'_o$ are denoted by L'_o and U'_o, respectively.

The return of asset j in period t relative to period $t-1$ is not known with certainty, and it is assumed that a set $\mathbb{S}$ (indexed by s) of scenarios may occur. The occurrence probability of scenario s and the return of asset j in period t under scenario s are denoted by p_s and $r_{j,t,s}$, respectively.

Two objectives are considered. The first one, which is of greater importance, is to minimize the expected postponed obligations (failure to meet obligations). The second objective is to maximize the minimum of the asset-portfolio value under all scenarios at the end of the planning horizon. We denote the importance coefficients of these objectives by positive parameters μ_1 and μ_2, respectively.

In addition to the above statements, the following assumptions are made:

A1: New insurance policies which are added during the planning horizon are not taken into account.

A2: The transaction costs associated with buying and selling of assets are neglected.

A3: When discussing the insurance premium, we assume that the administrative expenses have already been deducted.

A4: The life table, biometric parameters, and the number of surrender requests are assumed to be deterministic.

A5: In each period, insurance premiums are collected from policyholders at the beginning of the period, and then the obligations are fulfilled.

A6: At the beginning of the planning horizon, the insurer does not have any postponed obligation.

With respect to the above explanation, the sets, indices and parameters of the problem are defined in Table 1.

Table 1: Sets, indices and parameters

Notation	Description
$\mathbb{T} = \{1, \ldots, T\}$	The set of time periods (indexed by t) associated with the planning horizon. The indices $t = 0, -1, -2, \ldots$ are used to refer to periods before the planning horizon, and indices $t = T+1, T+2, \ldots$ denote periods after that.
$\mathbb{G} = \{1, \ldots, G\}$	The set of insurance policy groups (indexed by g), where policies within each group are similar in terms of insured persons age, death coverage premium, investment premium, death benefit, policy start period, and maturity period.
π_g	Death coverage premium associated with each policy of group $g \in \mathbb{G}$.
π'_g	Investment premium associated with each policy of group $g \in \mathbb{G}$.
d_g	Death benefit associated with each policy of group $g \in \mathbb{G}$.
τ_g	Start period associated with each policy of group $g \in \mathbb{G}$.
τ'_g	Maturity period associated with each policy of group $g \in \mathbb{G}$.
$r'_{g,t}$	The technical interest rate for each insurance policy of group g in period t specified according to the regulations ($g \in \mathbb{G}, t \in \mathbb{T} : t > 1$).
α	A parameter in the range (0,1) used to determine the participation profit. Indeed, if the realized return exceeds the technical interest rate, in addition to the technical interest rate, a portion α of the excess is credited to the policys reserve as the participation return.
$n_{g,t}$	The number of in-flow policies in group g who pay premiums at the beginning of period t ($g \in \mathbb{G}, t \in \mathbb{T}$).
$n'_{g,t}$	The number of persons of group g who die during period t ($g \in \mathbb{G}, t \in \mathbb{T}$).
$n''_{g,t}$	The number of persons of group g who surrender their policies during period t ($g \in \mathbb{G}, t \in \mathbb{T}$).
f_g	The investment reserve of group $g \in \mathbb{G}$ at the beginning of the planning horizon.
$e_{g,0}, e_{g,t}$	The total death benefit coverage reserves corresponding to all policies in group $g \in \mathbb{G}$ that the insurer must have at the beginning of the planning horizon and at the end of period $t \in \mathbb{T}$, respectively.
$e'_{g,t}$	The minimum investment reserve that each policy of group g must have at the end of period t ($g \in \mathbb{G}, t \in \mathbb{T}$).

Continued on next page

Table 1: Continued from previous page

Notation	Description
α'	The proportion of the accumulated investment reserve that the insurer pays to the policyholder upon surrender.
$\mathbb{J}$	The set of assets (such as bank deposits, units of fixed-income funds, stocks, and gold), indexed by j.
c_j	The capital allocated to asset $j \in \mathbb{J}$ at the beginning of the planning horizon.
L_j, U_j	Lower and upper limits on the proportion of the portfolio value allocated to asset j, specified by the regulations.
$\mathbb{O}$	The index set associated with given subsets of assets, indexed by o.
$\mathbb{J}'_o$	A subset of $\mathbb{J}$ including the assets, the total investment to which is required to fall within a given range specified by the regulations.
L'_o, U'_o	Lower and upper limits on the proportion of the portfolio value allocated to the assets of the subset $\mathbb{J}'_o$ with respect to regulations.
$\mathbb{S}$	The set of scenarios, indexed by s.
p_s	The occurrence probability of scenario s.
$r_{j,t,s}$	The return of asset j in period t related to period $t-1$ under scenario s ($t \in \mathbb{T}, t > 1, s \in \mathbb{S}$).
$\beta_{s,s',t}$	The binary parameter that is 1 if when making the decisions of period t, the scenarios s and s' are indistinguishable with respect to the realized information, otherwise 0 ($t \in \mathbb{T}, s, s' \in \mathbb{S}$).
μ_1, μ_2	Importance coefficients associated with the first and the second objectives, respectively.

3 Optimization model

Decision variables are defined as follows:

$x^{\text{Buy}}_{j,t,s}$ Nonnegative continuous variable indicating the amount of asset j, purchased in period t under scenario s, expressed in monetary units ($j \in \mathbb{J}, t \in \mathbb{T}, s \in \mathbb{S}$).

$x^{\text{Sell}}_{j,t,s}$ Nonnegative continuous variable indicating the amount of asset j, sold in period t under scenario s, expressed in monetary units ($j \in \mathbb{J}, t \in \mathbb{T}, s \in \mathbb{S}$).

$x^{\text{Hold}}_{j,t,s}$ Nonnegative continuous variable indicating the amount of money allocated to asset j in the asset-portfolio in period t (after all transactions) under scenario s, expressed in monetary units ($j \in \mathbb{J}, t \in \mathbb{T}, s \in \mathbb{S}$).

$w_{t,s}$ Free continuous variable indicating the return rate of the insurer's investment-portfolio in period t relative to period $t-1$ under scenario s ($t \in \mathbb{T}, t > 1, s \in \mathbb{S}$).

$q_{t,s}$ Nonnegative continuous variable indicating the amount of shortfall (postponed obligation) in period t under scenario s ($t \in \mathbb{T} \cup \{0\}, s \in \mathbb{S}$).

$h_{g,t,s}$ Nonnegative continuous variable representing the return rate credited to the investment reserve of group g in period $t-1$ under scenario s ($g \in \mathbb{G}, t \in \mathbb{T} : t > 1, s \in \mathbb{S}$).

$y_{g,t,s}$ Nonnegative continuous variable indicating the investment reserve of each policy of group g in period t under scenario s ($g \in \mathbb{G}, t \in \mathbb{T}, s \in \mathbb{S}$).

v Nonnegative continuous variable indicating the minimum value of asset-portfolio the end of the planning horizon over all scenarios

The sequence of decisions made during the planning horizon is as follows: In the first period, when the exact return rate of each asset in the second period relative to the first period is still unknown, decisions related to the asset-portfolio adjustment (i.e., $x^{Buy}_{j,1,s}, x^{Sell}_{j,1,s}$, and $x^{Hold}_{j,1,s}$) which we refer to as first-stage decisions are made. Then, the uncertainty in the return rate of each asset in the second period relative to the first period is realized, and accordingly, the value of variables $w_{2,s}, h_{g,2,s}$, and $y_{g,2,s}$ is determined and the portfolio adjustment decisions (i.e., $x^{Buy}_{j,2,s}, x^{Sell}_{j,2,s}$, and $x^{Hold}_{j,2,s}$) and the postponed obligations (i.e., $q_{2,s}$) are decided. Therefore, the variables $w_{2,s}, h_{g,2,s}, y_{g,2,s}, x^{Buy}_{j,2,s}, x^{Sell}_{j,2,s}, x^{Hold}_{j,2,s}$, and $q_{2,s}$ constitute the second-stage decisions. Next, the uncertainty in the return rate of each asset in the third period relative to the second period is realized, and accordingly, decisions $w_{3,s}, h_{g,3,s}, y_{g,3,s}, x^{Buy}_{j,3,s}, x^{Sell}_{j,3,s}, x^{Hold}_{j,3,s}$, and $q_{3,s}$, which we refer to as the third-stage decisions, are made. The same process is repeated. Finally, the Tth-stage decisions (i.e., $w_{T,s}, h_{g,T,s}, y_{g,T,s}, x^{Buy}_{j,T,s}, x^{Sell}_{j,T,s}, x^{Hold}_{j,T,s}$, and $q_{T,s}$) are made when the uncertainty in the return rate of each asset in the Tth period relative to the $(T-1)$th period is realized. Therefore, we deal with a T-stage stochastic programming problem.

The problem if formulated as the following nonlinear programming (NLP) model which we refer to as multi-stage stochastic ALM (MSALM) model. **MSALM**

$$\min z = \mu_1 \left(\sum_{s \in \mathbb{S}} p_s \left(\sum_{t \in \mathbb{T}} q_{t,s} \right) \right) - \mu_2 v \tag{4}$$

s.t.

$$c_j + x^{\text{Buy}}_{j,1,s} - x^{\text{Sell}}_{j,1,s} = x^{\text{Hold}}_{j,1,s} \quad \forall j \in \mathbb{J}, s \in \mathbb{S} \tag{5}$$

$$(1 + r_{j,t,s})\, x^{\text{Hold}}_{j,t-1,s} + x^{\text{Buy}}_{j,t,s} - x^{\text{Sell}}_{j,t,s} = x^{\text{Hold}}_{j,t,s} \quad \forall j \in \mathbb{J},\ t \in \mathbb{T} : t > 1,\ \forall s \in \mathbb{S} \tag{6}$$

$$L_j \sum_{j' \in \mathbb{J}} x^{\text{Hold}}_{j',t,s} \le x^{\text{Hold}}_{j,t,s} \le U_j \sum_{j' \in \mathbb{J}} x^{\text{Hold}}_{j',t,s} \quad \forall j \in \mathbb{J},\ t \in \mathbb{T},\ s \in \mathbb{S} \tag{7}$$

$$L'_o \sum_{j' \in \mathbb{J}} x^{\text{Hold}}_{j',t,s} \le \sum_{j \in \mathbb{J}'_o} x^{\text{Hold}}_{j,t,s} \le U'_o \sum_{j' \in \mathbb{J}} x^{\text{Hold}}_{j',t,s} \quad \forall o \in \mathbb{O},\ t \in \mathbb{T},\ s \in \mathbb{S} \tag{8}$$

$$w_{t,s} \sum_{j \in \mathbb{J}} x^{\text{Hold}}_{j,t-1,s} = \sum_{j \in \mathbb{J}} r_{j,t,s} x^{\text{Hold}}_{j,t-1,s} \quad \forall t \in \mathbb{T} : t > 1,\ \forall s \in \mathbb{S} \tag{9}$$

$$-\sum_{j \in \mathbb{J}} x^{\text{Hold}}_{j,t-1,s} \le w_{t,s} \le \sum_{j \in \mathbb{J}} x^{\text{Hold}}_{j,t-1,s} \quad \forall t \in \mathbb{T} : t > 1,\ \forall s \in \mathbb{S} \tag{10}$$

$$h_{g,t,s} \ge r'_{g,t} \quad \forall g \in \mathbb{G},\ \forall t \in \mathbb{T} : 1 < t \le \tau'_g,\ \forall s \in \mathbb{S} \tag{11}$$

$$h_{g,t,s} \ge r'_{g,t} + \alpha(w_{t,s} - r'_{g,t}) \quad \forall g \in \mathbb{G}, t \in \mathbb{T} : 1 < t \le \tau'_g,\ \forall s \in \mathbb{S} \tag{12}$$

$$y_{g,1,s} = f_g \quad \forall g \in \mathbb{G},\ s \in \mathbb{S} \tag{13}$$

$$y_{g,t,s} = (1 + h_{g,t,s})(y_{g,t-1,s} + \pi'_g) + \frac{(h_{g,t,s} - r'_{g,t})e_{g,t}}{n_{g,t}}$$

$$\forall g \in \mathbb{G},\ \forall t \in \mathbb{T} : 1 < t \le \tau'_g,\ \forall s \in \mathbb{S} \tag{14}$$

$$q_{0,s} = 0 \quad \forall s \in \mathbb{S} \tag{15}$$

$$\sum_{j \in \mathbb{J}} x^{\text{Sell}}_{j,t,s} + \sum_{g \in \mathbb{G}:\tau'_g \ge t} n_{g,t}(\pi_g + \pi'_g) + q_{t,s} = \sum_{j \in \mathbb{J}} x^{\text{Buy}}_{j,t,s} + \sum_{g \in \mathbb{G}:\tau'_g = t} (n_{g,t} y_{g,t,s} + n'_{g,t} d_g) +$$

$$\sum_{g \in \mathbb{G}:\tau'_g > t} n'_{g,t}(d_g + y_{g,t,s}) + \sum_{g \in \mathbb{G}:\tau'_g > t} \alpha' n''_{g,t} y_{g,t,s} + q_{t-1,s} \quad \forall t \in \mathbb{T}, s \in \mathbb{S} \tag{16}$$

$$v \le \sum_{j \in \mathbb{J}} x^{\text{Hold}}_{j,T,s} \quad \forall s \in \mathbb{S} \tag{17}$$

$$x^{\text{Buy}}_{j,t,s} = x^{\text{Buy}}_{j,t,s'} \quad \forall j \in \mathbb{J}, t \in T, s, s' \in \mathbb{S} : s < s' \text{ and } \beta_{s,s',t} = 1$$

$$x^{\text{Sell}}_{j,t,s} = x^{\text{Sell}}_{j,t,s'} \quad \forall j \in \mathbb{J}, t \in \mathbb{T}, s, s' \in \mathbb{S} : s < s' \text{ and } \beta_{s,s',t} = 1$$

$$x^{\text{Hold}}_{j,t,s} = x^{\text{Hold}}_{j,t,s'} \quad \forall j \in \mathbb{J}, t \in \mathbb{T}, s, s' \in \mathbb{S} : s < s' \text{ and } \beta_{s,s',t} = 1$$

$$w_{t,s} = w_{t,s'} \quad \forall t \in \mathbb{T} : t > 1, \forall s, s' \in \mathbb{S} : s < s' \text{ and } \beta_{s,s',t} = 1$$

$$q_{t,s} = q_{t,s'} \quad \forall t \in \mathbb{T}, s, s' \in \mathbb{S} : s < s' \text{ and } \beta_{s,s',t} = 1$$

$$h_{g,t,s} = h_{g,t,s'} \quad \forall g \in \mathbb{G}, t \in \mathbb{T} : t > 1, \forall s, s' \in \mathbb{S} : s < s' \text{ and } \beta_{s,s',t} = 1$$

$$y_{g,t,s} = y_{g,t,s'} \quad \forall g \in \mathbb{G}, t \in \mathbb{T}, \forall s, s' \in \mathbb{S} : s < s' \text{ and } \beta_{s,s',t} = 1 \tag{18}$$

$$x^{\text{Buy}}_{j,t,s}, x^{\text{Sell}}_{j,t,s}, x^{\text{Hold}}_{j,t,s} \ge 0 \quad \forall j \in \mathbb{J}, t \in \mathbb{T}, s \in \mathbb{S} \tag{19}$$

$$w_{t,s} \text{ free} \quad \forall t \in \mathbb{T} : t > 1, \forall s \in \mathbb{S} \tag{20}$$

$$q_{t,s} \ge 0 \quad \forall t \in \mathbb{T} \cup \{0\}, \forall s \in \mathbb{S} \tag{21}$$

$$h_{g,t,s} \ge 0 \quad \forall g \in \mathbb{G}, t \in \mathbb{T} : t > 1, \forall s \in \mathbb{S} \tag{22}$$

$$y_{g,t,s} \ge 0 \quad \forall g \in \mathbb{G}, t \in \mathbb{T}, s \in \mathbb{S} \tag{23}$$

$$v \ge 0 \tag{24}$$

Constraints (5) and (6) determine the capital allocated to each asset after buying and selling transactions for each period under each scenario. Constraints (7) and

(8) ensure the satisfaction of the lower and upper limits specified in the insurance regulations. Constraints (9) and (10) are linear restatements of the following constraint calculating the return rate of the insurer's investment-portfolio in period t under scenario s:

$$w_{t,s} = \frac{\sum_{j\in\mathbb{J}} r_{j,t,s} x^{\text{Hold}}_{j,t-1,s}}{\sum_{j\in\mathbb{J}} x^{\text{Hold}}_{j,t-1,s}} \quad \forall t \in \mathbb{T} : t > 1, \forall s \in \mathbb{S} \tag{25}$$

Constraint (10) ensures that if no portfolio is formed in period $t-1$ (i.e., $\sum_{j\in\mathbb{J}} x^{\text{Hold}}_{j,t-1,s} = 0$), then $w_{t,s}$ takes zero.

Constraints (11) and (12) are linear restatements of the following constraint, determining the participation return rate credited to each policy of group g in period t under scenario s:

$$h_{g,t,s} \geq \max(r'_{g,t}, \alpha w_{t,s}) \quad \forall g \in \mathbb{G}, t \in \mathbb{T}, s \in \mathbb{S} \tag{26}$$

The investment reserve of group g at the beginning of the planning horizon is determined by constraint (13). Constraint (14) represents the growth of the investment reserve for each policy of each group, where the first part applies the realized return to the investment reserve, and the second part applies the difference between the realized return and the guaranteed return to the death coverage reserve.

Constraint (15) sets the postponed obligations to zero at the beginning of the planning horizon (Assumption A6). Constraint (16) represents the cash flow in each period, where the left-hand side includes the total money obtained by asset selling and the insurance premiums received from in-flow policies. The right-hand side shows total payments (including reserves paid upon policy maturity, death benefits and reserves paid upon insured persons death, and reserves paid for surrendered policies). The variables $q_{t,s}$ and $q_{t-1,s}$ denote the amount of postponed obligations in period t and $t-1$, respectively, under scenario s.

Two objectives are considered. The first one which is of greater importance, minimizes the total expected shortfall (postponed obligations) during the planning horizon. It is formulated as follows:

$$\min \sum_{s\in\mathbb{S}} p_s \left(\sum_{t\in\mathbb{T}} q_{t,s} \right)$$

The second objective, formulated as below, maximizes the minimum asset-portfolio value at the end of the planning horizon under all scenarios:

$$\max_{s\in\mathbb{S}} \min_{j\in\mathbb{J}} \sum_{j\in\mathbb{J}} x^{\text{Hold}}_{j,T,s}$$

Considering the importance coefficients μ_1 and μ_2, we get a single-objective problem with the following objective function:

$$\min \left(\mu_1 \sum_{s\in\mathbb{S}} p_s \left(\sum_{t\in\mathbb{T}} q_{t,s} \right) \right) - \mu_2 \min_{s\in\mathbb{S}} \sum_{j\in\mathbb{J}} x^{\text{Hold}}_{j,T,s}$$

To linearize the above function, the term $\min_{s\in\mathbb{S}}\sum_{j\in\mathbb{J}}x^{\text{Hold}}_{j,T,s}$ is substituted by the nonnegative continuous variable v, and constraints (17) and (24) are included.

Constraint set (18) contains non-anticipativity constraints indicating that if scenarios s and s' are indistinguishable when making decisions in period t, those decisions should be the same under both scenarios. Finally, constraints (19)-(24) determine the types of variables.

4 Computational results

In this section, the performance of the proposed model is evaluated over different instances. Experiments are carried out on a laptop running Windows 10 with a Core$^{\text{TM}}$i7 processor and 16 GB of RAM. The model is implemented in the Pyomo package included in Python, and the solver Lindoglobal is utilized in its default setting to solve the model. The stopping criteria used to solve the model is either achieving a relative gap of 10^{-6} or reaching the time limit of 3600 seconds, whichever occurs first.

4.1 Characteristics of datasets

The proposed model is evaluated over several artificially generated datasets under the assumptions of insurance industry of Iran We assume that the set of assets includes bank deposits, units of fixed-income investment funds, Islamic securities, stocks (two market indices are considered as representatives of stocks), and gold. Planning is conducted annually over T future periods. The return rates associated with bank deposits, fixed-income investment funds, and government Islamic securities are assumed to be constant across all scenarios and estimated based on historical data and ARIMA prediction model. For the return rates associated with two market indices and gold, two scenarios are generated using ARIMA and AR prediction models. Hence, eight scenarios with equal occurrence probabilities are considered. The investment-portfolio must comply with the conditions, taken from Regulation No. 104 of the Supreme Insurance Council, specified in Table 2.

For the number of insurance groups, three cases 5, 10, and 20 are considered, and the insured persons age, insurance premium, and other information related to groups are generated based on real data. Parameters α and α' are set at 0.85 and 0.9, respectively, concerning regulations specified by the insurance of Iran. The value of f_g is fixed at $(1+r^{'}_{g,1})e^{'}_{g,0}$, and the total value of the asset-portfolio at the beginning of the planning horizon is assumed to be $\sum_{g\in\mathbb{G}}\Big((1+r^{'}_{g,1})(e_{g,0}+n_{g,1}e^{'}_{g,0})\Big)$, from which, the proportions 0.5, 0.1, and 0.4 are invested in bank deposits, Islamic securities, and stocks, respectively. Parameters μ_1 and μ_2 are set at 100 and 1, correspondingly.

Table 2: Lower and upper limits on the amount of investment, taken from Regulation No. 104 of the Supreme Insurance Council

Asset group	Limits
Bank deposits and units of fixed-income investment fund	At least 15% and at most 60% of the portfolio value. Note: Total investment in fixed-income funds cannot exceed 9% of the portfolio value. Additionally, the investment in any single fund cannot exceed 3% of the portfolio value.
Islamic securities	At least 10% and at most 20% of the portfolio value. Note: At least 50% of the investments specified in this item must be allocated to government Islamic securities.
Stocks	At most 60% of the portfolio value
Gold fund	At most 3% of the portfolio value

4.2 Results

The results are reported in Table 3 for instances with $G = 5, 10, 20$ and $T = 3, 5$. The first two columns of this table show the number of groups and periods. The column labeled by z_{MSALM} represents the objective value associated with the solution obtained by the solver within a time-limit of 3600 seconds. Further, considering LB as the lower bound on the optimal objective value reported by the solver, the column labeled by Gap shows the relative gap between z_{MSALM} and LB, calculated as $\frac{(z_{\text{MSALM}} - \text{LB})}{\text{LB}} \times 100$, evaluating the quality of the solution returned by the solver. The column labeled by Time indicates the time taken by the solver (in seconds). The symbol > 3600, placed in some cells of this column, indicates that the solver stops due to the given time-limit.

To assess the importance of the stochastic model, we use a metric, namely value-of-stochastic-solution and refer to it as VSS_1. To calculate this metric, the expected value of each uncertain parameter is computed as $\bar{r}_{j,t} = \sum_{s \in \mathbb{S}} p_s r_{j,t,s}$. Then, the deterministic model corresponding to expected values is solved. Considering $x^{*\text{Buy}}_{j,1}$, $x^{*\text{Sell}}_{j,1}$, and $x^{*\text{Hold}}_{j,1}$ as the first-stage decisions in the optimal solution to the deterministic model, some constraints fixing the first-stage decisions at $x^{*\text{Buy}}_{j,1}$, $x^{*\text{Sell}}_{j,1}$, and $x^{*\text{Hold}}_{j,1}$ are included in the MSALM model. This restricted model is solved and its optimal objective value is denoted by z_{EEV_1}. The metric VSS_1, calculated by $\text{VSS}_1 = \frac{z_{\text{EEV}_1} - z_{\text{MSALM}}}{|z_{\text{EEV}_1}|} \times 100$, shows the percentage of improvement achieved by the MSALM model in comparison to the deterministic model.

In all instances provided in Table 3, no postponed obligation occurred in any period, and the negative numbers expressed in z_{MSALM} represent the value of $-\min_{s \in \mathbb{S}} \sum_{j \in \mathbb{J}} x^{\text{Hold}}_{j,T,s}$ (in monetary units). As observed in Table 3, the metric VSS_1 has taken the value 17.3%, on average, indicating that using the stochastic model instead of the deterministic one can increase the portfolio value at the end of the planning horizon by 17.3%. Additionally, the results show that with an increase in the number of periods and groups, the resolution time of the proposed model

Table 3: Results

Dataset	G	T	No. Const.	No. Var.	z_{MSALM}	Gap(%)	Time(s)	z_{EEV_1}	VSS_1 (%)
1	5	3	1589	682	-118262.2	0.00	29.2	-97531.1	17.5
2	10	3	1993	866	-192672.9	0.00	46.5	-158147.2	17.9
3	15	3	2413	1066	-332397.5	0.00	29	-274214.7	17.5
4	20	3	2833	1266	-445529.5	0.62	> 3600	-366935.7	17.6
5	5	5	2213	1130	-224520.6	2.31%	> 3600	-187184.9	16.6
6	10	5	2825	1442	-314706.7	10.26	> 3600	-260044.1	17.4
7	15	5	3485	1802	-582504.7	1.45	> 3600	-485250	16.7
8	20	5	4145	2162	-727787.7	5.27	> 3600	-604248.7	17.0
Ave.									17.3

increases. However, our observations indicate that for instances interrupted due to the given time limit, the solver has found the best solution within the first 100 seconds, and the remaining time has been spent on improving the lower bound and proving the optimality. To accelerate the resolution process, suggesting an efficient method based on the decomposable structure of the problem would be a valuable direction for future research.

The findings indicate that relying solely on deterministic model may lead to suboptimal outcomes, particularly under market volatility or when facing uncertain events. The VSS highlights the advantage of the stochastic model over the deterministic one, showing that accounting for uncertainty in parameters can significantly improve portfolio value. Hence, insurers using the stochastic model could potentially achieve a better allocation of assets, as it is better equipped to handle the inherent uncertainties.

5 Conclusions

In this paper, a new multi-stage stochastic programming model was presented for the ALM problem related to companies providing with-profit life insurance policies. The proposed model was evaluated on different datasets and the VSS metric confirmed the importance of incorporating uncertainty. Given that the resolution time of the model increases with the number of scenarios, groups, and time periods, proposing an efficient algorithm that utilizes the decomposable structure of the problem is suggested as a future work. Additionally, a valuable research direction would be to extend the model to consider how the performance of the investment-portfolio affects the policyholders likelihood of surrendering. Another significant area for future research is to extend the model to handle uncertainties related to whether premiums are paid on time.

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