Journal of Mathematics and Modeling in Finance (JMMF)
Vol. 5, No. 1, Winter & Spring 2025
Research paper

Option pricing under non-normal distribution in mixed of Gram-Charlier model and fractional models (A case study of Iran Stock Exchange)

Mohammad Reza Haddadi[1], **Hossein Nasrollahi**[2]

[1] Zagros Financial Research Group, Ayatollah Boroujerdi University, Boroujerd, Iran.
haddadi@abru.ac.ir

[2] Zagros Financial Research Group, Ayatollah Boroujerdi University, Boroujerd, Iran.
hosein.nasrolahi@abru.ac.ir

Abstract:
In order to reduce the risk of financial markets, various tools have emerged, and option contracts are the most common tools in this regard. The Black-Scholes model is used to price a wide range of options contracts. The basic assumption in this model is to follow the normal distribution of returns. But the reality of the market indicates the skewness and kurtosis of the data, which reduces the accuracy of calculating the option price. The Gram-Charlie model has more flexibility than Black-Scholes model with abnormal skewness and kurtosis. The main purpose of this research is to determine the European call option price using non-normal data. In this regard, we present new models, fractional Gram-Charlier model and mixed fractional Gram-Charlier model, for option pricing. For this purpose, the data of Shasta and Khodro symbols have been selected from Iran Stock Exchange that Khodro in the period 2020-07-27 to 2023-11-1 and Shasta in the period 2022-7-25 to 2023-11-1 have been used. The results of this research show that Shasta has more abnormal skewness and kurtosis than Khodro. The option price calculated with the Gram-Charlier and extended models of Gram-Charlier are shown a smaller error compared to other models in the Shasta. Also, the results show that under abnormal skewness and kurtosis, our new models have more flexibility than the Black-Scholes model and fractional models.

Keywords: Black-Scholes model, fractional Brownian motion model, Gram-Charlier expansion, Option pricing, Stochastic volatility;
Classification: MSC2010 or JEL Classifications: 47H08, 47H10, 45B05.

1 Introduction

The most famous model for the valuation of European options is called the Black-Scholes model, which was presented in 1973. The Black-Scholes model assumes that stock returns follow a normal distribution with constant volatility. Hull (1993) and Natenberg (1994) point out that stock returns show non-normal skewness and kurtosis, and volatility deviations are the result of empirical violations of the normality assumption [12, 17]. Despite its advantages, such as simplicity and having

[1]Corresponding author
Received: 05/12/2024 Accepted: 06/03/2025
https://doi.org/10.22054/JMMF.2025.83277.1154

an explicit form for the transaction price, this model has always faced criticism due to unrealistic assumptions. The fundamental premise of the Black-Scholes model is that the future price distribution of the underlying assets follows a log-normal pattern. However, in actual financial markets, the price dynamics of assets exhibit a heavier tail than that of log-normal. In this way, we will face different distributions of the prices of buying or selling options, the difference of which is in the tail of the distribution. On the other hand, in the Black-Scholes model, the stock price volatility is considered constant, while the empirical results show that the underlying asset price volatility is non-constant. Also, in the Black-Scholes model, the constancy of volatility is considered as the main assumption, while it became clear over time, this assumption does not apply to options that have different exercise prices. Volatility smile and asymmetric volatility were famous phenomena that violated this assumption. These phenomena have been proposed in the form of abnormal behavior in asset return rate distribution. In fact, the distribution of the real fluctuation, in comparison with the bell curve, has a normal, asymmetric, skewed distribution and also has a long-term memory.

One of the major problems we face in financial economics is Brownian motion, which is considered a source of uncertainty and is used in option pricing. A stochastic process B_τ is a Brownian motion if the following condition holds:
(i) For $\tau > s$; $v > u$ and $u > \tau$, $B_\tau - B_s$ and $B_v - B_u$ are independent.
(ii) For all $\tau > s$; $B_\tau - B_s \sim N(0, \tau - s)$.
(iii) $B_0 = 0$.

In spite of allure universal request in various positions, practical evidence has abandoned to substantiate Brownian motion as the beginning of uncertainty. Cause return distributions noticed in fiscal markets do not trail the Gaussian law and occasion succession of return distributions exhibit complete dependency.

To overcome the interpretation, various weighty tailed distributions have happened evaluated as attainable opportunities in the writing of return distribution and partial Brownian motion has happened introduced to capture the general reliance of financial data.

To address this interpretation, several heavy-tailed distributions have been assessed as viable options for modeling return distributions. Additionally, partial Brownian motion has been introduced to capture the dependencies commonly observed in financial time series.

Fractional Brownian motion is a generalization of standard Brownian motion acquired by adjoining individual parameter, named the Hurst parameter, that was introduced in 1968 by Mandelbrot and Van Ness [14]. On the other hand, fractional Geometric Brownian Motion is a generalization of Geometric Brownian Motion (GBM). GBM is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion. It is an important example of stochastic processes satisfying a SDE. GBM was used in the BlackScholes model. Fractional Brownian motion is an extention of Brownian motion. Unlike classical Brownian motion, the increments of Fractional Brownian motion need not be independent [21].

The Brownian dynamics in stochastic volatility have gradually been supplanted by more elaborate Gaussian processes, particularly those exhibiting long-range dependence. (Comte and Renault, 1998) [4]. This is true for fractional Brownian motion when $H > 0.5$. See also Breidt et al. (1998) [3] and Sottinen (2001) [22] among many others.

Definition 1.1. Let H be a constant belonging to $(0, 1)$. A fractional Brownian motion (GFBM), B^H_τ, with the Hurst index H is a continuous and centred Gaussian process with covariance function:

$$E(B^H_\tau B^H_s) = \frac{1}{2}(\tau^{2H} + s^{2H} - |\tau - s|^{2H})$$

that
(i) $B^H_0 = 0$ and $E(B^H_\tau) = 0$ for every $\tau \geq 0$.
(ii) For $\tau > s$, $v > u$ and $u > \tau$, the increments $B_\tau - B_s$ and $B_v - B_u$ are independent.
(iii) For all $\tau > s$; $B_\tau - B_s \sim N(0, |\tau - s|^{2H})$.
(iv) B^H, has continuous trajectories.

The Hurst exponent details the the quality of being rough on the surface of the resultant motion, accompanying a greater value leading to a more flowing motion. If $H = 0.5$ therefore the process is really a Brownian motion; if $H > 0.5$ so the increments of the process are definitely equated; if $H < 0.5$ therefore the increments of the process are negatively correlated [22]. Fractional Brownian motion has found applications in modeling random processes appearing in economics, finance, hydrology, wave propagation in random media, etc [21].

Barunik et al. (2012) apply the genealized Hurst parameter to multifractal analysis [1]. Gu et al. (2012), Meng and Wang (2010), and Xiao et al. (2010) regard fractional Brownian motion as a fundamental diffusive process [8, 15, 23]. Most of the literature focuses on arbitrage and its exclusion in fractional Brownian motion models. Rogers (1997), used fractional Brownian motion within financial models. Bender (2003) and Dasgupta and Kallianpur (2000) made an explicit arbitrage strategy within the fractional market [2,5]. Cheridito (2003) successfully developed arbitrage strategies within both the fractional Bachelier model and the fractional Black-Scholes model [9]. In continuation, Hu and Øksendal (2003) gave the result of a fractional BlackScholes model excluding arbitrage provided [11]. They showed that the correct usage of fractional Brownian motion inherently implies dynamic market incompleteness (see also Della Ratta et al., 2008, [6]).

Necula (2002) give the following formula for the European call option by fractional Brownian motion [19]:

$$C(\tau, T) = S_\tau N(\zeta_1) - K e^{-r(T-\tau)} N(\zeta_2), \tag{1}$$

where

$$\zeta_1 = \frac{\ln(\frac{S_\tau}{K}) + r(T-\tau) + \frac{1}{2}\sigma^2(T^{2H} - \tau^{2H})}{\sigma\sqrt{T^{2H} - \tau^{2H}}},$$

$$\zeta_2 = \frac{\ln(\frac{S_\tau}{K}) + r(T-\tau) - \frac{1}{2}\sigma^2(T^{2H} - \tau^{2H})}{\sigma\sqrt{T^{2H} - \tau^{2H}}}.$$

Rostek (2009) gave a formula for pricing European options using conditional expectation [20]. Also, Rostek and Schöbel gave a note on the use of fractional Brownian motion for financial modeling [21]. Ghasemifard et al. (2022) considered option valuation in markets with finite liquidity under fractional CEV assets [7]. Nasiri et al. (2022) gave a numerical method for solving the underlying price problem driven by a fractional Levy process [18].

In the following give the asset price dynamics that follows geometric Brownian motion (GBM) and geometric fractional Brownian motion (GFBM) [?]. A stochastic process X_t is said to follow a geometric Brownian motion (GBM) if the asset price evolves according to the following dynamics:

$$dX_t = \mu X_t dt + \sigma X_t dB_t, \tag{2}$$

where B_t is a Brownian motion, μ is the constant drift, and σ is the constant volatility. The solution to Eq. (2) for any chosen initial value X_0 is expressed as follows:

$$X_t = X_0 \exp(\mu t - \frac{1}{2}\sigma^2 t + \sigma B_t).$$

Also, stochastic process S_t follows a GFBM if the asset price follows the following dynamics:

$$dX_t = \mu X_t dt + \sigma X_t dB_t^H, \tag{3}$$

where B_t^H is an FBM with $H \in (0,1)$, μ is the constant drift, and r is the constant volatility. By using the Wick Ito Skorohod integrals for GFBM, the solution to Eq. (3) for any chosen initial value X_0 is expressed as follows

$$X_t = X_0 \exp(\mu t - \frac{1}{2}\sigma^2 t^{2H} + \sigma B_t^H).$$

2 Materials and Methods

2.1 Hurst parameter estimation

A number of rough approaches were suggested for figuring out the parameter H. These non-parametric methods are primarily useful as illustrative tools and are particularly chosen to provide an initial estimate of H. They are still less acceptable for statistical conclusion, as, for most of these methods, it is not easy to get confidence intervals.

Aggregate variance technique

An important property of long-memory processes is that the variance of the sample mean converges to 0 than $\frac{1}{n}$, that N is dimension of the sample. Therefore, it can be inferred that $var(\bar{X}_n) \approx \epsilon n^{2H-2}$, that $\epsilon > 0$ and $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. This consideration allows to introduce the following procedure to estimate H. Hence

$$H \approx 1 + \frac{1}{2}(slope). \tag{4}$$

For short-range dependence or independence among the observations, the slope is equal to 1.

Least square regression

In the frequency domain, analyzing time series is simply the examination of a stationary process using its spectral representation at the origin. Hence the least square regression in the spectral domain exploits:

$$g(\zeta) \approx c_g|\zeta|^{1-2H},$$

An estimator for the spectral density function $g(\zeta)$ is the periodogram, given by

$$I(\zeta_j) = \frac{1}{2\pi n}|\sum_{s=1}^{n}(X_t - \bar{X}_n)e^{is\zeta_j}|^2 = \frac{1}{2\pi}\sum_{k=-(n-1)}^{n-1}\bar{\gamma}(k)e^{ik\zeta_j}$$

that $\zeta_j = 2\pi j/n$ are the Fourier frequencies. For long-memory processes, the following result can be demonstrated:

$$\log I(\zeta_{j,n}) \approx \log c_f + (1-2H)\log\zeta_{j,n} + \log\xi_j \tag{5}$$

that ξ_j are independent standard exponential random variables. Hence $1-2H$ and so H can be estimated by least square regression.

Maximum Likelihood Estimators

Aggregate variance and least squares regression are the primary heuristic methods for estimating H. While these methods are effective for assessing the presence of long-memory, they are inadequate for analyzing short-term properties and are also unsuitable for statistical inference. Maximum Likelihood is a possible alternative approach is to use parametric models for estimating H, that assumes a functional form for the spectral density $g(\zeta)$ and considers minimizing parameters based upon specific assumptions about the model.

Maximum Likelihood Estimators are more efficient; however, calculating them precisely presents computational challenges. This is due to the potential numerical instability in evaluating the inverse and determinant of the variance matrix, and the number of operations required increases with the square of the dataset's dimension.

2.2 Mixed models for European call option

I. Gram-Charlier expansion model

The Gram-Charlier expansion, if properly used (Hung et al., 2015 [10]), allows the generation of distributions with the desired volatility, skewness and kurtosis with μ_3, μ_4, μ_5 are skewness, kurtosis, and super skewness respectively.

$$BSG{=}BS + \mu_3 \mathcal{Q}_3 + (\mu_4 - 3)\mathcal{Q}_4, \tag{6}$$

which BS is the call option price of BlackScholes model, i.e.

$$BS{=}S_0 \mathrm{N}\left(\zeta\right) - Ke^{-rT}\mathrm{N}\left(\zeta - \sigma\sqrt{T}\ \right),$$

$$\zeta{=}\frac{\ln\left(\frac{S_0}{K}\right) + \left[r + \frac{1}{2}\left(\sigma^2\right)\right]T}{\sigma\sqrt{T}},$$

$$\mathcal{Q}_3{=}\frac{1}{3!}\ S_0\sigma\sqrt{T}(\left(2\sigma\sqrt{T} - \zeta\right)n\left(\zeta\right) + \sigma^2 TN\left(\zeta\right)),$$

and

$$\mathcal{Q}_4{=}\frac{1}{4!}\ S_0\sigma\sqrt{T}\left(\left(\zeta^2 - 1 - 3\sigma\sqrt{T}\left(\zeta - \sigma\sqrt{T}\right)\right)n\left(\zeta\right) + \sigma^3 T^{\frac{3}{2}}N(\zeta)\right).$$

Also, the model of Gram-Charlier expansion by super skewness:

$$BCGS{=}BS + \mu_3\mathcal{Q}_3 + \left(\mu_4 - 3\right)\mathcal{Q}_4 + \left(\mu_5 - 10\mu_3\right)\mathcal{Q}_5, \tag{7}$$

which

$$\mathcal{Q}_5{=}\frac{1}{120}\ S_0\sigma\sqrt{T}\left(\sigma^4T^2N\left(\zeta\right) + N(\zeta)\left[4\sigma^3T^{\frac{3}{2}} - 6\zeta\sigma^2T + 3\zeta^2\sigma\sqrt{T} + \zeta\sigma\sqrt{T} - 3\sigma\sqrt{T} - \zeta^3 + 3\zeta\right]\right).$$

II. Fractional Gram-Charlier expansion model

The formula that Rostek derives is given as follows [20]

$$BS_f(t,T) = S_tN(\zeta_1^H) - Ke^{-r(T-t)}N(\zeta_2^H), \tag{8}$$

where

$$\zeta_1^H = \frac{\ln(\frac{S_t}{K}) + r(T-t) + \frac{1}{2}\rho_H\sigma^2(T-t)^{2H}}{\sqrt{\rho_H}\sigma(T-t)^H}$$

and

$$\zeta_2^H = \frac{\ln(\frac{S_t}{K}) + r(T-t) - \frac{1}{2}\rho_H\sigma^2(T-t)^{2H}}{\sqrt{\rho_H}\sigma(T-t)^H}.$$

Hence, in the following we give the mixed model of Gram-Charlier expansion and formula Rostek

$$BSG_f{=}BS_f + \mu_3\mathcal{Q}_3 + \left(\mu_4 - 3\right)\mathcal{Q}_4 \tag{9}$$

Also, the mixed model of Gram-Charlier expansion and Rostek formula by super skewness:

$$BSGS_f{=}BS_f + \mu_3\mathcal{Q}_3 + \left(\mu_4 - 3\right)\mathcal{Q}_4 + \left(\mu_5 - 10\mu_3\right)\mathcal{Q}_5. \tag{10}$$

III. Mixed fractional Gram-Charlier expansion model

Let H be a constant belonging to $(0,1)$. An mixed fractional Brownian motions of parameter H, a and b is a stochastic process $M^H=\{M_t^H, t\geq 0\}$ define as follows

$$M_t^H = aB_t^H + bB_t. \tag{11}$$

In the following we give and consider some special cases of the mixed fractional Brownian motion of (11). If $a=b=\frac{1}{\sqrt{2}}$ then (11) is the mixed fractional Brownian motion (Mliki, 2023 [?]) for which the European call option is given by

$$BS_m{=}S_0\mathrm{N}\left(\zeta_1\right) - Ke^{-rT}\mathrm{N}\left(\zeta_2\right)$$

where

$$\zeta_1 = \frac{\ln(\frac{S_t}{K}) + r(T-t) + \frac{1}{2}\sigma^2(T^{2H}-t^{2H}) + \frac{1}{2}\sigma^2(T-t)}{\sqrt{\sigma(T^{2H}-t^{2H}) + \frac{1}{2}\sigma^2(T-t)}}$$

and

$$\zeta_2 = \frac{\ln(\frac{S_t}{K}) + r(T-t) - \frac{1}{2}\sigma^2(T^{2H}-t^{2H}) - \frac{1}{2}\sigma^2(T-t)}{\sqrt{\sigma(T^{2H}-t^{2H}) + \frac{1}{2}\sigma^2(T-t)}}.$$

Hence, in the following we give mixed model of Gram-Charlier expansion and formula Rostek

$$BSG_m{=}BS_m + \mu_3\mathcal{Q}_3 + (\mu_4-3)\,\mathcal{Q}_4. \tag{12}$$

Also, mixed fractional Brownian motion model and Gram-Charlier expansion and formula Rostek by super skewness:

$$BSGS_m{=}BS_m + \mu_3\mathcal{Q}_3 + (\mu_4-3)\,\mathcal{Q}_4 + (\mu_5-10\mu_3)\,\mathcal{Q}_5. \tag{13}$$

3 Results and Discussion

In this paper, we utilized daily data for the Shasta and Khodro symbols selected from the Iran Stock Exchange. The data for Khodro covers the period from July 27, 2020, to November 1, 2023, while the data for Shasta spans from July 25, 2022, to November 1, 2023. The closing price is the last price closed in 2023. Their simulation was obtained using matlab and R. The reason for choosing these two companies is the biggest companies in the Iran. It should be noted that the skewness and kurtosis of these two companies are different and they give in Table 1.

Table 1: Descriptive statistics of daily returns

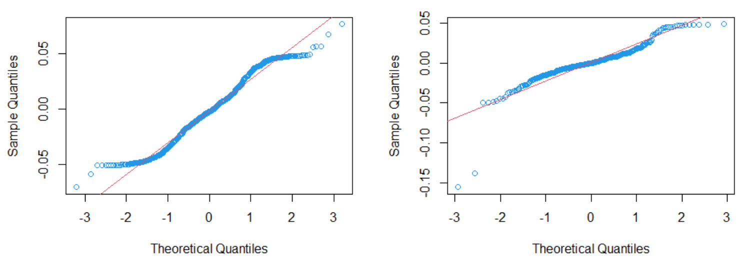
	kurtosis	skewness	super skewness	SD	mean
Khodro	2.17270	0.07925	0.44583	0.02872	0.0016
Shasta	13.64985	-1.5719	-75.1722	0.02309	0.0006

The normality test of the daily returns of Khodro and Shasta is given in Table 2. The results of both Jarque-bera and Shapiro-Wilk tests show that the data for the two stocks are non-normal. The next step is to draw the quantile-quantile diagram,

Table 2: Normality test of daily returns

Variable analysis	Jarque-bera		Shapiro-Wilk	
	X-squared	p-value	W	p-value
Khodro	21.138	0.00002	0.97383	0.00000
Shasta	1515.6	0.00000	0.86809	0.00000

which is presented in Figure 1.



(a) Normal Q-Q Plot Khodro (b) Normal Q-Q Plot Shasta

Figure 1: Normal QQ-plots

In the following, distribution selection is examined using information criteria.

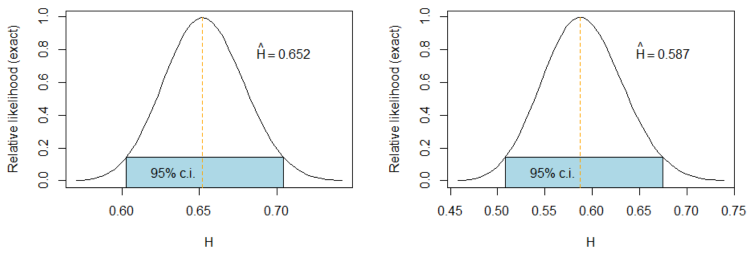
Table 3: select the optimal distribution

	Khodro		Shasta	
Criteria	AIC *	BIC **	AIC	BIC
nornal	-3087.102	-3077.929	-1359.629	-1352.289
snormal	-3086.733	-3072.975	-1358.951	-1347.941
STD	-3085.102	-3071.343	-1441.579	-1430.570
SSTD	-3084.725	-3066.380	-1440.800	-1426.120
GED	-3140.913	-3127.155	-1360.108	-1349.099
SGED	-3091.987	-3073.642	-1361.609	-1346.929

* Akaike Information Criterion
** Bayesian Information Criterionn

The model that has the lowest information criteria value is the most optimal.

From Table 3 Generalized Error distribution (GED) has the least value of the information criterias for Khodro and student-t distribution (STD) as the least value of the information criterias for Shasta.



(a) MLEHurst parameter Khodro (b) MLE Hurst parameter Shasta

Figure 2: Exact MLE estimation Hurst parameter

European call option price of Black-Scholes (BS) model, Gram-Charlier model (BSG) and Gram-Charlier model by super skewness ($BSGS$) with r = 0.23 of Khodro is given in Table 4.

Table 4: The comparison between the value of European call options

T	S	K	Market	BS		BCG		BCGS	
				price	% e	price	% e	price	% e
28	2319	2000	362	365.9378	1.09	364.6487	0.73	364.4219	0.67
28	2319	2400	96	99.25377	3.39	103.46992	7.78	103.67383	7.99
28	2319	2800	21	12.94990	38.33	11.88692	43.40	12.27243	41.56
56	23191	1800	633	589.4080	6.89	587.3591	7.21	587.2982	7.22
56	2319	2400	192	166.0433	13.52	172.0282	10.40	172.1133	10.36
56	2319	3000	71	23.64182	66.70	22.28358	68.61	23.02892	67.56
91	2319	1900	598	554.9489	7.20	554.5788	7.26	554.0341	7.35
91	2319	2400	312	235.2170	24.61	242.6619	22.22	242.5834	22.25
91	2319	3000	108	60.65129	43.84	61.78932	42.79	63.12465	41.55

European call option price of BS, BSG and $BSGS$ models with r = 0.23 of Shasta is given in Table 5.
In the short term, the Black-Scholes model has better answers than most of the existing models, even models with stochastic volatility (for instance Heston's model). But, according to the Table 4, in the short term, medium term and long term option prices are estimated with less error than the Black-Scholes model. The error of

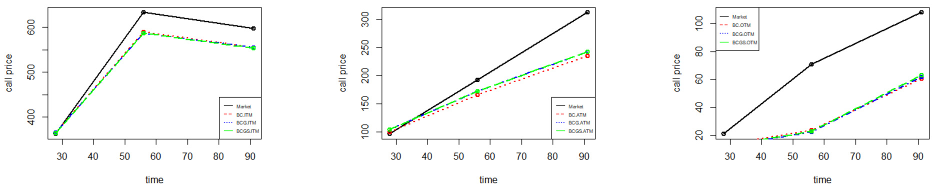
Maximum Likelihood Estimation

Table 5: The comparison between the value of European call options

T	S	K	Market	BS		BCG		BSGS	
				price	% e	price	% e	price	% e
35	1106	712	400	409.5316	2.38	409.2637	2.31	409.5313	2.38
35	1106	1112	44	59.30488	34.78	37.26692	15.30	28.90709	34.30
35	1106	1312	5	6.01506	20.30	7.63074	52.61	37.00496	640
63	1106	712	466	421.7409	9.50	422.5649	9.32	426.6728	8.44
63	1106	1112	1	86.30157	8530	57.92916	5692	39.59546	3859
63	1106	1312	40	19.36014	51.60	5.98926	85.03	71.52772	78.82
119	1106	512	656	630.9897	3.81	629.3970	4.05	629.6654	4.01
119	1106	1112	128	131.10958	2.43	94.90941	25.85	57.85335	54.80
119	1106	1512	54	16.60369	69.25	16.00521	70.36	101.50041	87.96

the option price in Shasta (Table 5) is more than that of Khodro (Table 4), which is due to the negative skewness in Shasta.

Due to the negative skewness in Shasta, the error has increased compared to Khodro. Also, from Table 4 it can be seen total errors ($E = \sum_{i=1}^{k} e_i^2$) of Khodro stock in ITM are $E_{BS} = 100.4$, $E_{BSG} = 105.2$ and $E_{BSGS} = 106.6$. On the other hand, from Table 5, total errors of Shasta stock in ITM are $E_{BS} = 110.4$, $E_{BSG} = 108.6$ and $E_{BSGS} = 93$. Hence error European call option price of BSG and $BSGS$ models is less than of BS model, and so they are better models.



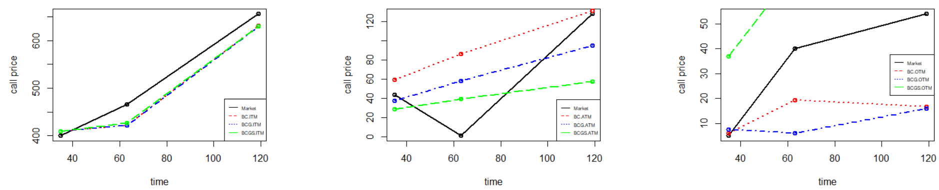
(a) ITM (b) ATM (c) OTM

Figure 3: Khodro call option price of BS, BSG, BSGS and Market

European call option price with fractional Black-Scholes (BS_f), , BSG_f and $BSGS_f$ models with r = 0.23 and H=0.652 of Khodro is given in Table 6.

European call option price with fractional Black-Scholes (BS_f), BSG_f and $BSGS_f$ models with r = 0.23 and H=0.587 of Shasta is given in Table 7.

In ATM, the BS_f model has less error in Khodro's option price than BSG_f and $BSGS_f$. On the other hands, the BS_f model has more error in Shasta's option price than BSG_f and $BSGS_f$.



(a) ITM (b) ATM (c) OTM

Figure 4: Shasta call option price of BS, BSG, BSGS and Market

Table 6: The comparison between the value of European call options

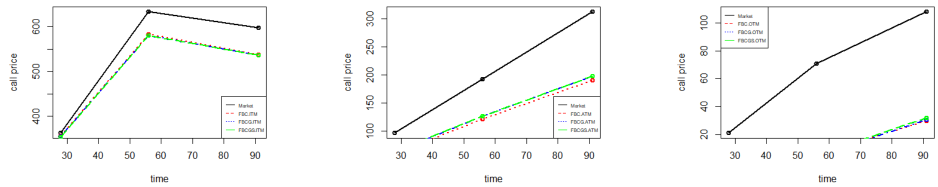
T	S	K	Market	BS_f		BSG_f		BSGS_f	
				Price	%Error	Price	%Error	Price	%Error
28	2319	2000	362	355.4096	1.82	354.1205	2.18	353.8938	2.24
28	2319	2400	96	58.73695	38.81	62.95309	34.42	63.15701	34.21
28	2319	2800	21	1.42589	93.21	0.36291	98.27	0.74842	96.43
56	2319	1800	633	582.5686	7.96	580.5197	8.29	580.4589	8.30
56	2319	2400	192	120.6450	37.16	126.6300	34.04	126.7151	34.00
56	2319	3000	71	5.86296	91.74	4.50471	93.655	5.25	92.60
91	2319	1900	598	536.9017	10.21	536.5317	10.27	535.9870	10.37
91	2319	2400	312	190.0439	39.09	197.4888	36.70	197.4104	36.72
91	2319	3000	108	29.58709	72.60	30.72512	71.55	32.06045	70.31

Table 7: The comparison between the value of European call options

T	S	K	Market	BS_f		BSG_f		$BSGS_f$	
				Price	% Error	Price	% Error	Price	% Error
35	1106	712	400	409.5311	2.38	409.2632	2.31	409.5308	2.38
35	1106	1112	44	49.86362	13.33	27.82567	36.76	19.46584	55.76
35	1106	1312	5	2.39722	52.05	4.01290	19.74	33.38713	567.74
63	1106	712	466	421.7145	9.50	422.5385	9.33	426.6464	8.44
63	1106	1112	1	76.6825	7568	48.31017	4731	29.97647	2897
63	1106	1312	40	12.39484	69.01	0	100	64.73	61.82
119	1106	512	656	630.9889	3.81	629.3962	4.05	629.6646	4.01
119	1106	1112	128	122.60296	4.22	86.40279	32.50	49.34673	61.45
119	1106	1512	54	11.45402	78.79	10.85554	79.90	96.35074	78.43

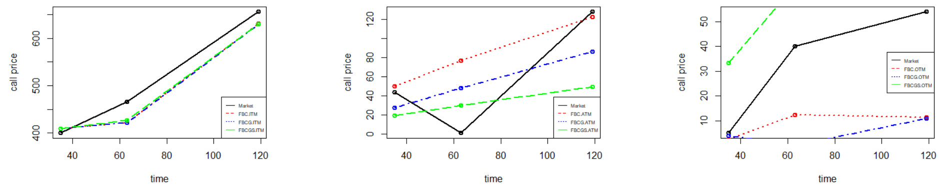
Also, from Table 6 it can be seen total errors of Khodro stock in ITM are $E_{BS_f} = 171.2$, $E_{BSG_f} = 179.1$ and $E_{BSGS_f} = 181.4$. On the other hand, from Table 7, that the total errors of Shasta in ITM are $E_{BS_f} = 110.5$, $E_{BSG_f} = 108.2$ and $E_{BSGS_f} = 107.9$. Hence error European call option price of BSG_f and $BSGS_f$ models is less than that of the of BS_f model, and so they are better models.

Comparing models BS, BSG and $BSGS$ with models BS_f, BSG_f and $BSGS_f$, respectively, the error in Shasta is not changed, while in Khodro, shows that the error is increased. Hence this means that there is no significant difference in symbols with skewness and kurtosis of fractional and non-fractional models of option pricing.



(a) ITM (b) ATM (c) OTM

Figure 5: Khodro call option price of BS_f, BSG_f, $BSGS_f$ and Market



(a) ITM (b) ATM (c) OTM

Figure 6: Shasta call option price of BS_f, BSG_f, $BSGS_f$ and Market

European call option price with mixed of BS_m, BSG_m and $BSGS_m$ models with r = 0.23 and H=0.652 of Khodro were given in Table 8.

European call option price with BS_m, BSG_m and $BSGS_m$ models with r = 0.23 and H=0.587 of Shasta were given in Table 8.

Also, from Table 8 it can be seen total errors of Khodro stock in ITM are $E_{BS_m} = 44.2$, $E_{BSG_m} = 44.74$ and $E_{BSGS_m} = 44.74$. On the other hand, from Table 9, total errors of Shasta stock in ITM are $E_{BS_m} = 230.8$, $E_{BSG_m} = 220.7$ and $E_{BSGS_m} = 206.6$. Hence error European call option price of BSG_m and $BSGS_m$ models is less than of BS_m model, and so they are better models.

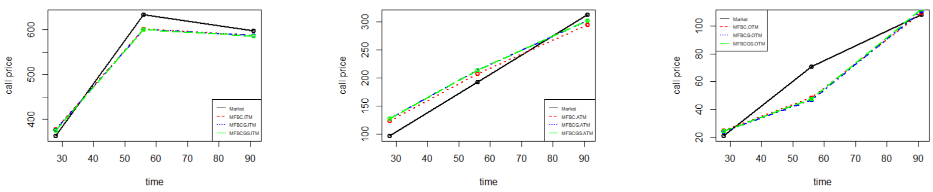
Therefore, it can be seen that Shasta, which has more skewness and kurtosis than Khodro, the option price error has increased more than 5 times. Also, based on the observations, it is concluded that methods 2 and 3 have a better answer than method 1. That is, in stocks with skewness and kurtosis, methods 2 and 3 have

Table 8: The comparison between the value of European call options

T	S	K	Market	BS_m		BSG_m		BSGS_m	
				Price	% Error	Price	% Error	Price	% Error
28	2319	2000	362	376.7804	4.08	375.4926	3.73	375.2669	3.67
28	2319	2400	96	123.24822	28.38	127.46437	32.77	127.66828	32.99
28	2319	2800	21	25.21250	20.06	24.14952	14.99	24.53504	16.83
56	2319	1800	633	602.0100	4.89	599.9611	5.22	599.9003	5.23
56	2319	2400	192	207.0553	7.84	213.0403	10.9	213.1254	11.00
56	2319	3000	71	48.40346	31.82	47.04522	33.74	47.79056	32.69
91	2319	1900	598	586.99	1.84	586.63	1.90	586.0852	1.99
91	2319	2400	312	293.8871	5.80	301.3320	3.42	301.2536	3.44
91	2319	3000	108	109.45361	1.34593	110.5916	2.40	111.9269	3.63

Table 9: The comparison between the value of European call options

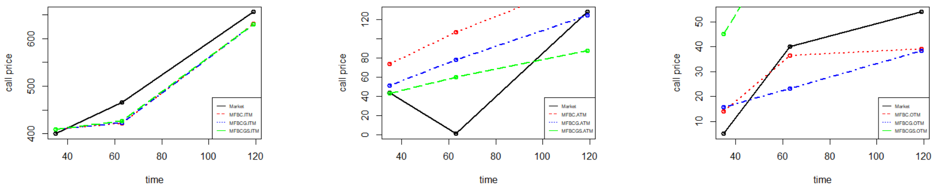
T	S	K	Market	BS_m		BSG_m		BSGS_m	
				Price	% Error	Price	% Error	Price	% Error
35	1106	712	400	409.5588	2.39	409.2909	2.32	409.5585	2.39
35	1106	1112	44	73.5610	67.18	51.5231	17.09	43.1633	1.90
35	1106	1312	5	14.0789	181.58	15.6946	213.89	45.0688	801.37
63	1106	712	466	422.1950	9.40	424.0190	9.22	427.1269	8.34
63	1106	1112	1	106.7970	10579	78.4246	7742	60.0909	5909
63	1106	1312	40	36.5079	8.73	23.2113	41.97	88.84	122.10
119	1106	512	656	631.0689	3.8	629.4762	4.04	629.7445	4.00
119	1106	1112	128	160.84372	25.65	124.6435	2.62	87.5875	31.57
119	1106	1512	54	39.0801	27.63	38.4816	28.74	123.9768	129.59



(a) ITM (b) ATM (c) OTM

Figure 7: Khodro call option price of BS_m, BSG_m, $BSGS_m$ and Market

better answers. For the share of Khodro, the combined method in Table 8 gives a better answer than the fractional and normal method in Tables 4 and 7. But for Shasta, which has skewness and kurtosis, according to Table 9, the error increases



(a) ITM (b) ATM (c) OTM

Figure 8: Shasta call option price of BS_m, BSG_m, $BSGS_m$ and Market

greatly.

4 Conclusion and Suggestions

Conclusion

Considering the fundamental role of financial markets in the economic development of any country, a detailed examination of these markets from different aspects seems necessary. Option trading is one of the trading tools introduced to the financial markets in order to reduce risk. In this regard, it is customary to use the Black-Scholes model for pricing a wide range of options contracts. In this model, the normality of data distribution is a basic assumption. The Gram-Charlie model has more flexibility than Black-Scholes model with abnormal skewness and kurtosis.

The main purpose of this research is to determine the European call option price using a non-normal data. In this regard, extended models of Gram-Charlier in fractional and mixed with titles $BSGS_f$ and $BSGS_m$ were presented.

The data of Shasta and Khodro symbols have been selected from Iran Stock Exchange that Khodro in the period 2020-07-27 to 2023-11-1 and Shasta in the period 2022-7-25 to 2023-11-1 have been used. The results of this research show that Shasta has more abnormal skewness and kurtosis than Khodro. The option price calculated with the Gram-Charlier and extended models of Gram-Charlier (i.e. $BSGS_f$ and $BSGS_m$) were shown a smaller error compared to other models in the Shasta. Also, the results of this research show that European option pricing by using $BSGS_m$ has more flexibility than the Black-Scholes, models of the fractional Brownian motion (for instance [21]), models of the mixed fractional Brownian motion (for instance [?]) and also $BSGS_f$ with abnormal skewness and kurtosis.

Suggestions

The following are some suggested ideas for future research, focusing on option pricing models:

- Create a hybrid model that combines the strengths of the Heston model with the flexibility of the Gram-Charlier model.
- Studying the effect of different levels of market volatility on the accuracy of the Gram Charlier model in option pricing.
- Comparing machine learning techniques with the Gram-Charlier model to improve option pricing accuracy.

Bibliography

[1] J. Barunik, T. Aste, T. Di Matteo, and R. Liu, *Understanding the source of multifractality in financial markets*, Physica A **391** (2012), no. 17, 4234–4251.

[2] C. Bender, *Integration with respect to fractional Brownian motion and related market models*, Ph.D. thesis, University of Konstanz, Department of Mathematics and Statistics, 2003.

[3] F. J. Breidt, N. Crato, and P. de Lima, *The detection and estimation of long memory in stochastic volatility*, J. Econom. **83** (1998), 325–348.

[4] F. Comte and E. Renault, *Long memory in continuous-time stochastic volatility models*, Math. Finance **8** (1998), 291–323.

[5] A. Dasgupta and G. Kallianpur, *Arbitrage opportunities for a class of Gladyshev processes*, Appl. Math. Optim. **41** (2000), 377–385.

[6] L. Della Ratta, F. Leccadito, and G. Urga, *The Fractional Merton Model: A New Approach to Credit Risk Pricing (Revised)*, Centre for Econometric Analysis London, Working Paper, 2008.

[7] A. Ghasemifard, S. Banihashemi, and A. Babaei, *Option valuation in markets with finite liquidity under fractional CEV assets*, J. Math. Model. Finance **2** (2022), no. 2, 167–180.

[8] H. Gu, J. Liang, and Y. Zhang, *Time-changed geometric fractional Brownian motion and option pricing with transaction costs*, Physica A **391** (2012), no. 15, 3971–3977.

[9] P. Cheridito, *Arbitrage in fractional Brownian motion models*, Finance Stoch. **7** (2003), no. 4, 533–553.

[10] S. Hung, H. Hsi, and S. Han, *Option pricing under truncated Gram-Charlier expansion*, N. Am. J. Econ. Finance **32** (2015), 77–97.

[11] Y. Hu and B. Øksendal, *Fractional white noise calculus and applications to finance*, Infinite Dimens. Anal. Quantum Probab. Relat. Top. **6** (2003), no. 1, 1–32.

[12] J. Hull, *Options, Futures, and Other Derivative Securities*, Prentice Hall, Englewood Cliffs, NJ, 1993.

[13] S. N. Iqmal Ibrahim, M. Misiran, and M. F. Laham, *Geometric fractional Brownian motion model for commodity market simulation*, Alex. Eng. J. **60** (2021), no. 1, 955–962.

[14] B. B. Mandelbrot and J. W. Van Ness, *Fractional Brownian motions, fractional noises and applications*, SIAM Rev. **10** (1968), 422–437.

[15] L. Meng and M. Wang, *Comparison of Black-Scholes formula with fractional Black-Scholes formula in the foreign exchange option market with changing volatility*, Asia-Pac. Financ. Mark. **17** (2010), no. 2, 99–111.

[16] E. Mliki, *Correlation structure of time-changed generalized mixed fractional Brownian motion*, Fractal Fract. **7** (2023), 591.

[17] S. Natenberg, *Option Volatility and Pricing*, Probus Publishing, Chicago, 1994.

[18] T. Nasiri, A. Zakeri, and A. Aminataei, *A numerical method for solving the underlying price problem driven by a fractional Levy process*, J. Math. Model. Finance **2** (2022), no. 1, 195–208.

[19] C. Necula, *Option Pricing in a Fractional Brownian Motion Environment*, Preprint, Academy of Economic Studies, Bucharest, 2002.

[20] S. Rostek, *Option Pricing in Fractional Brownian Markets*, Springer, 2009.

[21] S. Rostek and R. Schöbel, *A note on the use of fractional Brownian motion for financial modeling*, Econ. Model. **30** (2013), 30–35.

[22] T. Sottinen, *Fractional Brownian motion, random walks and binary market models*, Finance Stoch. **5** (2001), 343–355.

[23] W. Xiao, W. Zhang, X. Zhang, and Y. Wang, *Pricing currency options in a fractional Brownian motion with jumps*, Econ. Model. **27** (2010), no. 5, 935–942.

How to Cite: Mohammad Reza Haddadi[1], Hossein Nasrollahi[2], *Option pricing under non-normal distribution in mixed of Gram-Charlier model and fractional models (A case study of Iran Stock Exchange)*, Journal of Mathematics and Modeling in Finance (JMMF), Vol. 5, No. 1, Pages:47–62, (2025).