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Research paper

On the Importance of Copula Choice in the Reliability Evaluation of Dependent Stress-Strength Models

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Abstract:
Reliability assessment, vital in high-stakes engineering, often employs the stress-strength model. However, traditional models frequently assume independence between stress and strength, an assumption that can lead to inaccurate reliability estimates when dependence exists due to real-world factors. To address this, the current study proposes a dependent stress-strength model using copula theory, which flexibly models dependence by separating marginal and joint distributions. Four copula familiesFarlie-Gumbel-Morgenstern, Ali-Mikhail-Haq, Gumbel's bivariate exponential, and Gumbel-Hougaardare investigated for their ability to capture diverse dependency patterns. The Inverse Lomax distribution is utilized for both stress and strength marginals due to its suitability for heavy-tailed reliability data. The copula dependence parameter θ is estimated via conditional likelihood and Blomqvist's beta-based method of moments. The asymptotic distributions of these estimators are derived, and their performance is evaluated through extensive simulations. The research thoroughly examines how system reliability R changes with θ across various model configurations. Findings indicate that the Gumbel-Hougaard copula demonstrates the highest sensitivity of R to θ, effectively capturing a wide range of dependency strengths. This paper highlights the critical need to incorporate dependence in stress-strength models and offers practical guidance for copula selection, thereby enhancing the accuracy and robustness of reliability predictions in complex engineering systems. A practical examination of a real dataset is conducted to demonstrate the concept.

Keywords: Ali-Mikhail-Haq, Blomqvist's beta, Copula, Farlie-Gumbel-Morgenstern, Gumbel-Hougaard, Reliability.
Classification: 62N05, 62N01.

1 Introduction

The role of reliability analysis in engineering systems remains essential because it protects both the safety and operational performance of multiple industrial sectors including aerospace and automotive and civil infrastructure and electronics. Reliability engineering depends on the stress-strength model as a fundamental analysis method to evaluate component or system operational probability during specific performance requirements. This model compares the stress applied to a system with its inherent strength, and reliability is determined by the probability that the

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strength exceeds the stress. Church and Harris [3] were the first to introduce the stress-strength reliability framework in 1970. A substantial body of literature has focused on analyzing the reliability of the stress-strength model, especially when both variables are assumed to be independent and follow the same parametric family. See also, Wong [27] and Sengupta [22]. According to Jovanovi and Raji [16] stress-strength model uses gamma distribution for X and exponential distribution for Y.

In Jovanović and Rajić [15] examined the reliability estimation problem involving two independent random variables$X$ with geometric and Y with exponential distributional characteristics. Hu et al. [12] developed their study by working to calculate confidence intervals for stress-strength reliability parameters within the case of geometric and exponential distributed variables X and Y.

The stress and strength variables in practical situations present dependence because of common environmental elements or production methods or operational variables. Traditional reliability models base their assumptions on independent relationships between stress and strength variables which produces inaccurate and optimistic prediction results. Neglecting statistically dependent relationships between variables produces major inaccuracies when computing failure risks because it results in underprediction of failure probabilities which endangers system safety. Dependent stress-strength models emerged as a solution to this issue.

One approach to considering dependencies is through the use of bivariate distributions. In the field of stress- strength models, bivariate distributions can be used to model the joint distribution of stress and strength, which can then be used to calculate the probability of failure. For instance, Nadarajah [17] investigated the reliability of dependent stress-strength models using a bivariate Beta distribution, while Gupta et al. [6] examined it through a bivariate log-normal distribution. Recently, Xavier and Jose [28] presented a Bayesian estimation of the reliability of the dependent stress-strength model using the bivariate exponentiated half-logistic distribution.

Copulas represent an alternative approach that lets users model random variable dependencies through flexible methods. Copulas function by breaking joint distribution modeling from individual variable distributions thus providing advanced mechanisms to depict variable interdependencies. Using copulas provides researchers a method to detect intricate dependencies between variables that bivariate distributions cannot visualize which strengthens stress resistance model performance and accuracy. This model is particularly useful in assessing the probability of failure of components under varying conditions. Recent advancements in the application of copula functions have significantly enhanced the analysis of stress-strength models by allowing for the modeling of dependencies between random variables.

The copula function has been employed by many researchers. Domma and Giordano [4] derived a reliability by utilizing the Farlie-Gumbel-Morgenstern (FGM) copula and generalized FGM copula to model the dependence. Patil et al. [19]

examine how dependency affects the probability $R = P(Y < X)$ under exponential distributions and introduce copula-based approaches for estimating R. Hakamipour [8] introduces a novel copula-based method for analyzing multicomponent stress-strength reliability, offering valuable insights into the dependence structures between stress and strength variables. Hakamipour [9] examines the estimation of reliability for s-out-of-k multicomponent systems based on the Gumbel copula function under progressively censored samples. Hakamipour [10] introduces a novel statistical framework for analyzing dependent competing risks in progressive-stress accelerated life tests, where units' lifetimes follow a Gompertz distribution and dependence is modeled using the Gumbel copula.

According to Sklar's theorem [25] mentioned below, using copulas is an effective method for modeling the dependency between variables. For details on copulas, we refer to Nelsen [18].

Sklar's Theorem: for any joint distribution function H of random variables X and Y with respective marginal distribution functions $F(x)$ and $G(y)$, there exists a copula function C such that for all x, y in $(-\infty, \infty)$,

$$H(x, y) = C(F(x), G(y)).$$

Moreover, if both F and G are continuous, the copula C is unique. In cases where continuity does not hold, C is uniquely determined on the product of the ranges of F and G, i.e., $RangeF \times RangeG$.

Various types of copula functions exist, each with distinct characteristics and benefits. The most frequently utilized copulas are the Gaussian, Clayton, Gumbel, and Frank copulas. These copulas vary in their shapes and properties, making each one suitable for different dependence structures. The choice of correct copula function serves as a fundamental element that determines the precision of dependence representation as well as estimation accuracy. The analysis of dependent stress-strength models requires selecting appropriate copulas because their choice determines both dependence strength and tail behavior. However, the appropriate choice of copula function is typically unknown. The most suitable model can be identified based on the Akaike Information Criterion (AIC).

Copulas incorporate a parameter θ, referred to as the dependence parameter. In this study, we concentrate on the FGM copula, a member of the non-Archimedean family, characterized by a dependence parameter θ within the interval $[-1, 1]$. Furthermore, we consider three copulas from the Archimedean family, each defined over distinct ranges of the dependence parameter θ: the Ali-Mikhail-Haq (AMH) copula, Gumbel's bivariate exponential (Gumbel's BE) copula, and the Gumbel-Hougaard (GH) copula.

The dependence parameter θ lies within the interval $[-1, 1]$ for the AMH copula, $[0, 1]$ for Gumbel's BE copula, and $[1, \infty]$ for the GH copula. The corresponding ranges of Kendall's tau (τ) for the FGM, AMH, Gumbel's BE, and GH copulas are $[-0.22, 0.22]$, $[-0.1817, 0.3333]$, $[-0.4, 0]$, and $[0, 1]$ respectively [20]. The FGM and

AMH copula models are capable of capturing both positive and negative dependence structures, whereas Gumbel's BE copula captures only negative dependence, and the GH copula represents solely positive dependence.

In this paper, we examine the expression for R and its estimation in the context of four significant copula models that have non-identical Inverse Lomax margins. To estimate R, we adopt two methods: one utilizes conditional likelihood, while the other employs the method of moments based on Blomqvist's beta. For each of the four copulas, we analyze the effect of dependence on R by illustrating how R varies with θ.

An expression for the reliability is

$$R = P[Y < X] = \int P\big[Y < x|X = x\big] f(x)dx, \tag{1}$$

where $P\big[Y < x|X = x\big]$ indicates the conditional probability, and f refers to the probability density function (PDF) of the random variable X.

The Inverse Lomax (IL) Distribution, known for its flexibility and heavy tails, has garnered interest in statistical modeling, especially in areas such as finance, reliability engineering, and risk analysis. As a member of an inverted family of distributions, the IL Distribution is applicable in various scenarios where the failure rate is non-monotonic [24]. It serves as an alternative to several other distributions, including Gamma and Weibull, among others [23]. For more information, please refer to Jamilu et al. [13]. In reliability analysis, the stress-strength model describes the lifetime of a component whose random strength X is subjected to a random stress Y. Consequently, we examine the IL distribution as the marginal distribution for the random variables X and Y, along with their corresponding Cumulative Distribution Functions.

$$F(x; \zeta_1, \gamma) = (1 + \frac{\gamma}{x})^{-\zeta_1}, \qquad x > 0 \tag{2}$$

and

$$G(y; \zeta_2, \gamma) = (1 + \frac{\gamma}{y})^{-\zeta_2}, \qquad y > 0 \tag{3}$$

where $\zeta_1, \zeta_2 > 0$ are shape parameters and $\gamma > 0$ is scale parameter.

In Section 2, we analyze R as a function of θ and the parameters $(\zeta_1, \zeta_2, \gamma)$ associated with the margins of the four copulas. To investigate how the dependency between X and Y influences R, we plot the graph of R versus θ for various combinations of $(\zeta_1, \zeta_2, \gamma)$. An explicit formula for R could not be established for these copulas with IL marginal distributions. Consequently, we employ a Monte Carlo integration method to calculate R for specified values of the margin parameters and θ.

Given fixed values of $(\zeta_1, \zeta_2, \gamma)$ and θ, we randomly generate N values of X_i from the IL distribution with parameters (ζ_1, γ) which has a PDF $f(x) = \frac{\gamma\zeta_1}{x^2}(1+\frac{\gamma}{x})^{-\zeta_1-1}$, for $x > 0$, $\zeta_1 > 0$, $\gamma > 0$ and approximate the integral $\int_0^\infty g(x)f(x)dx$ by

$\frac{1}{N}\sum_{i=1}^{N} g(X_i)$. According to the law of large numbers, the estimated result approaches the true value as $N \to \infty$. We chose $N = 100000$ for our calculations. To examine the influence of θ on R, we plot a graph of R against θ. Using observations from n independent pairs $(X_1, Y_1), \ldots, (X_n, Y_n)$, we explore two methods for estimating θ and, consequently, R. Both methods adhere to what is commonly referred to in the literature as the 'two-stage estimation procedure' [14]. In the initial phase, the marginal parameters are estimated using the Maximum Likelihood Estimation (MLE) method based on their respective probability distributions. The first strategy for estimating the dependence parameter θ involves solving the score equation $\frac{d\ell}{d\theta} = 0$, where ℓ represents the conditional log-likelihood function constructed from the conditional density of Y given X. This is achieved through the Newton-Raphson iterative procedure, employing the previously estimated marginal parameters. Details of this method are provided in Section 3. Alternatively, a nonparametric estimation technique is applied for θ. Such methods often rely on inverting dependence measures like Spearman's rho (ρ), Kendall's tau (τ), or Blomqvist's beta (β). Nevertheless, for the specific copula models considered here, closed-form expressions for the population versions of Spearman's rho and Kendall's tau are either unavailable or their inversion is analytically intractable. Therefore, we adopt an approach based on Blomqvist's beta, as suggested by Nelson [18] and originally introduced by Blomqvist [1]; further elaboration is given in Section 4. The asymptotic behavior of each estimator of R is examined within their respective sections. Section 5 reports the results of Monte Carlo simulations conducted to assess and compare the estimators' performance. To illustrate the influence of dependence on the estimation, we also include plots of R estimates as a function of the true parameter θ. In Section 6, we apply our findings to real data. Finally, conclusions are provided in Section 7.

2 Expressions for R

In the following subsections, we derive expressions for R corresponding to various copula families with IL marginal distributions. The form of R outlined in (1) for IL marginals is expressed as follows:

$$R = P[Y < X] = \int_0^\infty P\big[Y \le x | X = x\big] f(x) dx, \tag{4}$$

where $f(x) = \frac{\gamma\zeta_1}{x^2}(1 + \frac{\gamma}{x})^{-\zeta_1 - 1}$, for $x > 0$, $\zeta_1 > 0$, $\gamma > 0$ is PDF of X.

It can be noted that in the case of independence between the random variables X and Y, the following relation holds:

$$R = P[Y < X] = \frac{\zeta_1}{\zeta_1 + \zeta_2}.$$

If the joint distribution function of (X, Y) is represented as $C_\theta(F(x), G(y))$ using the copula function C_θ, then:

$$P\big[Y \leq y|X = x\big] = \frac{\partial C_\theta(u,v)}{\partial u}\ |_{u=F(x),v=G(y)}, \tag{5}$$

see Nelsen [18].

2.1 FGM Copula

The FGM copula [18] is given by

$$C_\theta(u,v) = uv[1+\theta(1-u)(1-v)]; \qquad 0 \leq u,v \leq 1;\ -1 \leq \theta \leq 1. \tag{6}$$

The bivariate copula serves as the joint distribution function for two random variables that have uniform margins. We will refer to these random variables as U and V throughout the paper. Therefore,

$$\frac{\partial C_\theta(u,v)}{\partial u} = v[1+\theta(1-2u)(1-v)].$$

Consequently, from 5, we obtain:

$$P\big[Y \leq y|X = x\big] = (1+\frac{\gamma}{y})^{-\zeta_2}\Big[1+\theta\big(1-(1+\frac{\gamma}{y})^{-\zeta_2}\big)\big(1-2(1+\frac{\gamma}{x})^{-\zeta_1}\big)\Big], \quad x>0,\ y>0. \tag{7}$$

Consequently, the expression for reliability R as presented in Equation (4) becomes:

$$R = \frac{\zeta_1}{\zeta_1+\zeta_2} + \theta\Big(\frac{2\zeta_1}{\zeta_1+\zeta_2} - \frac{2\zeta_1}{2\zeta_1+\zeta_2} - \frac{\zeta_1}{\zeta_1+2\zeta_2}\Big). \tag{8}$$

2.2 AMH Copula

The AMH copula [18] is defined as

$$C_\theta(u,v) = \frac{uv}{1-\theta(1-u)(1-v)}; \qquad 0 \leq u,v \leq 1;\ -1 \leq \theta \leq 1. \tag{9}$$

Then

$$\frac{\partial C_\theta(u,v)}{\partial u} = \frac{v\big(1-\theta(1-v)\big)}{\big(1-\theta(1-u)(1-v)\big)^2}.$$

Therefore, based on Equation (5),

$$P\big[Y \leq y|X = x\big] = \frac{(1+\frac{\gamma}{y})^{-\zeta_2}\Big(1-\theta\big(1-(1+\frac{\gamma}{y})^{-\zeta_2}\big)\Big)}{1-\theta\big(1-(1+\frac{\gamma}{x})^{-\zeta_1}\big)\big(1-(1+\frac{\gamma}{y})^{-\zeta_2}\big)}, \quad x>0,\ y>0. \tag{10}$$

Consequently, the reliability R indicated in Equation (4) is

$$R = \int_0^\infty \frac{\gamma\zeta_1}{x^2}\frac{(1+\frac{\gamma}{x})^{-(\zeta_1+\zeta_2)-1}\big(1-\theta+\theta(1+\frac{\gamma}{x})^{-\zeta_2}\big)}{1-\theta+\theta(1+\frac{\gamma}{x})^{-\zeta_2}+\theta(1+\frac{\gamma}{x})^{-\zeta_1}-\theta(1+\frac{\gamma}{x})^{-\zeta_1-\zeta_2}}dx. \tag{11}$$

2.3 Gumbel's BE Copula

Gumbel's BE copula [18] is given by

$$C_\theta(u,v) = u + v - 1 + (1-u)(1-v)e^{-\theta \ln(1-u)\ln(1-v)}; \qquad 0 \le u,v \le 1;\ 0 \le \theta \le 1. \tag{12}$$

Then,

$$\frac{\partial C_\theta(u,v)}{\partial u} = 1 - e^{-\theta \ln(1-u)\ln(1-v)}(1-v)\big(1-\theta\ln(1-v)\big).$$

So, from (5),

$$\begin{aligned} P\big[Y \le y|X=x\big] \quad &= \quad 1 - \big(1-(1+\frac{\gamma}{x})^{-\zeta_1}\big)^{-\theta\ln\left(1-(1+\frac{\gamma}{y})^{-\zeta_2}\right)}\big(1-(1+\frac{\gamma}{y})^{-\zeta_2}\big) \\ &\quad \Big(1-\theta\ln\big(1-(1+\frac{\gamma}{y})^{-\zeta_2}\big)\Big) \end{aligned} \tag{13}$$

Thus, the reliability R given in (4) is given by

$$\begin{aligned} R \quad &= \quad 1 - \int_0^\infty \frac{\gamma\zeta_1}{x^2}(1+\frac{\gamma}{x})^{-\zeta_1-1}\big(1-(1+\frac{\gamma}{x})^{-\zeta_2}\big)^{1-\theta\ln\left(1-(1+\frac{\gamma}{x})^{-\zeta_1}\right)} \\ &\quad \times\Big(1-\theta\ln\big(1-(1+\frac{\gamma}{x})^{-\zeta_2}\big)\Big)dx. \end{aligned} \tag{14}$$

2.4 GH Copula

This group of copulas was initially introduced by Émile and Gumbel [5] and is also mentioned by Hougaard [11]. It is defined as follows:

$$C_\theta(u,v) = \exp\big[-\big(-\ln u^\theta + -\ln v^\theta\big)^{\frac{1}{\theta}}\big]; \qquad 0 \le u,v \le 1;\ 1 \le \theta < \infty. \tag{15}$$

Therefore,

$$\frac{\partial C_\theta(u,v)}{\partial u} = \frac{-\ln u^{\theta-1}}{u}\Big(-\ln u^\theta + -\ln v^\theta\Big)^{\frac{1}{\theta}-1}\exp\big[-\big(-\ln u^\theta + -\ln v^\theta\big)^{\frac{1}{\theta}}\big].$$

Hence, from (5),

$$\begin{aligned} P\big[Y \le y|X=x\big] \quad &= \quad \exp\Big[-\big(\zeta_1^\theta\ln(1+\frac{\gamma}{x})^\theta + \zeta_2^\theta\ln(1+\frac{\gamma}{y})^\theta\big)^{\frac{1}{\theta}}\Big] \\ \times\zeta_1^{\theta-1}(1+\frac{\gamma}{x})^{\zeta_1}\ln(1+\frac{\gamma}{x})^{\theta-1}&\Big[\zeta_1^\theta\ln(1+\frac{\gamma}{x})^\theta + \zeta_2^\theta\ln(1+\frac{\gamma}{y})^\theta\Big]^{\frac{1}{\theta}-1}. \end{aligned} \tag{16}$$

Thus, the reliability R given in (4) is given by

$$R = \frac{\zeta_1^\theta}{\zeta_1^\theta + \zeta_2^\theta}. \tag{17}$$

2.5 Variation in R with Respect to θ

To explore the effect of dependence on R, we plot its variation with respect to the parameter θ for selected values of $(\zeta_1, \zeta_2, \gamma)$ across the four copula models. Set 1: $(\zeta_1, \zeta_2, \gamma) = (1, 2, 1)$, Set 2: $(\zeta_1, \zeta_2, \gamma) = (2, 1, 1)$, Set 3: $(\zeta_1, \zeta_2, \gamma) = (0.5, 1.5, 2)$, Set 4: $(\zeta_1, \zeta_2, \gamma) = (1.5, 0.5, 2)$, Set 5: $(\zeta_1, \zeta_2, \gamma) = (0.3, 0.7, 0.5)$, Set 6: $(\zeta_1, \zeta_2, \gamma) = (0.7, 0.3, 0.5)$. Figure 1 shows how R changes in relation to θ for the four copulas.

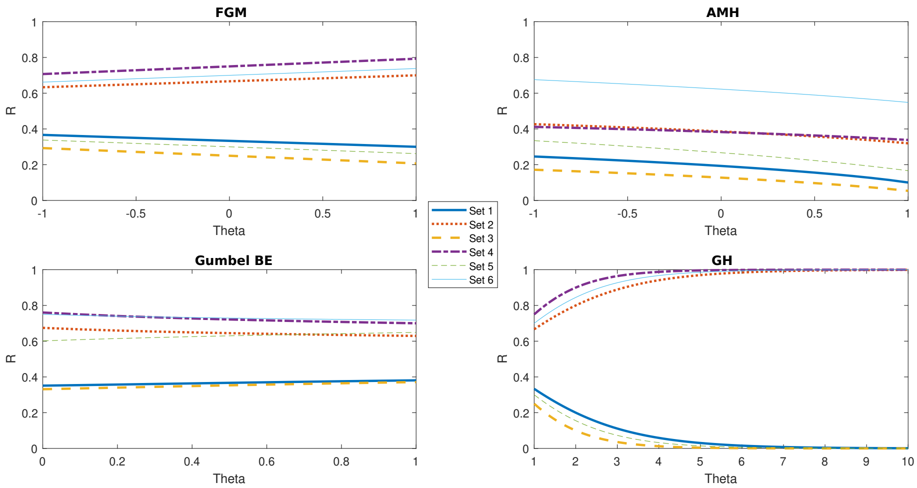

Figure 1: Variation in R against dependence parameter θ for the six sets.

Based on the graph, we can infer that if $\zeta_1 > \zeta_2$, then $R > 0.5$ for the FGM, Gumbel's BE, and GH copulas. Additionally, under the condition that $\zeta_1 > \zeta_2$, $R < 0.5$ tends to increase with increasing θ for the FGM, Gumbel's BE, and GH copulas, whereas it declines for the AMH copula. Conversely, if $\zeta_1 < \zeta_2$, then $R < 0.5$ for the FGM and GH copulas, and R decreases with θ for the FGM, AMH, and GH copulas.

Throughout the analysis, the graph of R generally appears to be nearly linear in relation to θ, with the exception of the GH copula. The changes in R concerning θ occur more rapidly for the GH copula compared to the other three copulas.

The graphical analysis demonstrates that the GH copula induces the most substantial variation in the reliability measure R as the dependence parameter θ changes, resulting in the widest observed range of R values across all copulas examined. In contrast, the Gumbel's BE copula yields minimal variability in R. Moreover, the difference between reliability estimates under independence and those obtained under dependence is relatively small for the FGM, AMH, and Gumbel's BE copulas. This limited sensitivity may be attributed to the narrower range of dependence these models are capable of capturing. In comparison, the GH copula supports a broader spectrum of dependency strength, as reflected in its higher achievable values of Kendall's tau and Blomqvist's beta, as discussed in the introduction. Accordingly, the reliability measure R exhibits greater sensitivity to changes in dependence when modeled using the GH copula.

3 Likelihood-Based Estimation of θ and R

Let $(x_i, y_i), i = 1, 2, \ldots, n$ represent a random sample drawn from the joint distribution function $H(x, y)$. To estimate the parameters, we adopt a two-step estimation procedure as outlined in [14]. In the initial stage, the marginal parameters ζ_1, ζ_2 and γ are obtained by maximizing the corresponding marginal likelihoods. As a result, the likelihood function associated with the observed values $(x_1, x_2, \ldots, x_n)$ is expressed as:

$$L(x_1, x_2, \ldots, x_n; \zeta_1, \gamma) = \gamma^n \zeta_1^n \prod_{i=1}^{n} \frac{1}{x_i^2}(1 + \frac{\gamma}{x_i})^{-\zeta_1 - 1},$$

and the log-likelihood function is given by

$$\begin{aligned} \ell &= \ln L(x_1, x_2, \ldots, x_n; \zeta_1, \gamma) \\ &= n \ln \gamma + n \ln \zeta_1 - 2 \sum_{i=1}^{n} \ln x_i - (\zeta_1 + 1) \sum_{i=1}^{n} \ln(1 + \frac{\gamma}{x_i}). \end{aligned}$$

Using the log-likelihood mentioned above, the MLEs $\hat{\zeta}_1$ and $\hat{\gamma}$ for ζ_1 and γ are determined as the solutions to

$$\frac{\partial \ell}{\partial \zeta_1} = \frac{n}{\zeta_1} - \sum_{i=1}^{n} \ln(1 + \frac{\gamma}{x_i}) = 0. \tag{18}$$

and

$$\frac{\partial \ell}{\partial \gamma} = \frac{n}{\gamma} - (\zeta_1 - 1) \sum_{i=1}^{n} \frac{1}{x_i + \gamma} = 0. \tag{19}$$

Considering the non-linearity of equations (18) and (19), the MLEs $\hat{\zeta}_1$ and $\hat{\gamma}$ of ζ_1 and γ are obtained using the Newton-Raphson iterative procedure. Likewise, the MLE $\hat{\zeta}_2$ for ζ_2 can be derived from the sample $(y_1, y_2, \ldots, y_n)$ drawn from Y.

Since the joint density is given by $h(x_i, y_i) = f(y_i|x_i) f_X(x_i)$ and $f_X(x_i)$ does not depend on θ, we maximize $\prod_{i=1}^{n} f(y_i|x_i)$ to estimate θ. We replace the parameters of the margins with their estimates. Let $L(y|x) = \prod_{i=1}^{n} f(y_i|x_i)$. The form of $\frac{\partial \ln L(y|x)}{\partial \theta}$, derived from the conditional distribution of $y_1, y_2, \ldots, y_n$ given $x_1, x_2, \ldots, x_n$, for each of the four copulas under consideration, is provided in the subsequent subsections.

3.1 FGM Copula

Based on Equation (7), the conditional probability density function of Y given $X = x$ is expressed as follows:

$$f(y|X=x) = \frac{\gamma\zeta_2}{y^2}(1+\frac{\gamma}{x})^{-\zeta_1}(1+\frac{\gamma}{y})^{-2\zeta_2-1} \times\Big((1+\frac{\gamma}{x})^{\zeta_1}(1+\frac{\gamma}{y})^{\zeta_2}+\theta\big((1+\frac{\gamma}{x})^{\zeta_1}-2\big)\big((1+\frac{\gamma}{y})^{\zeta_2}-2\big)\Big). \tag{20}$$

Therefore,

$$\frac{\partial \ln L(y|x)}{\partial\theta} = \sum_{i=1}^{n} \frac{\big((1+\frac{\gamma}{x_i})^{\zeta_1}-2\big)\big((1+\frac{\gamma}{y_i})^{\zeta_2}-2\big)}{(1+\frac{\gamma}{x_i})^{\zeta_1}(1+\frac{\gamma}{y_i})^{\zeta_2}+\theta\big((1+\frac{\gamma}{x_i})^{\zeta_1}-2\big)\big((1+\frac{\gamma}{y_i})^{\zeta_2}-2\big)} \tag{21}$$

We replace $(\zeta_1, \zeta_2, \gamma)$ with their estimates and solve $\frac{\partial \ln L}{\partial\theta} = 0$ using the Newton-Raphson method to estimate θ.

3.2 AMH Copula

The conditional density function of Y given that $X = x$, based on Equation (10), is expressed as follows:

$$f(y|X=x) = \frac{\gamma\theta\zeta_2(1+\frac{\gamma}{x})^{-\zeta_1}(1+\frac{\gamma}{y})^{-2\zeta_2-1}\big(1-\theta(1-(1+\frac{\gamma}{y})^{-\zeta_2})\big)}{y^2W^2} + \frac{\gamma\theta\zeta_2(1+\frac{\gamma}{y})^{-2\zeta_2-1}+\gamma\zeta_2(1+\frac{\gamma}{y})^{-\zeta_2-1}\big(1-\theta(1-(1+\frac{\gamma}{y})^{-\zeta_2})\big)}{y^2W}, \tag{22}$$

where

$$W = 1-\theta\big(1-(1+\frac{\gamma}{x})^{-\zeta_1}\big)\big(1-(1+\frac{\gamma}{y})^{-\zeta_2}\big).$$

and with the margin parameters replaced by their estimated values

$$\frac{\partial \ln L(y|x)}{\partial\theta} = -\sum_{i=1}^{n}\frac{1-(1+\frac{\gamma}{x_i})^{-\zeta_1}(1-(1+\frac{\gamma}{y_i})^{-\zeta_2})}{W_i} \sum_{i=1}^{n}\frac{2\theta-2+(1+\frac{\gamma}{x_i})^{-\zeta_1}\big(1-2\theta-2\theta(1+\frac{\gamma}{y_i})^{-2\zeta_2}\big)}{1-2\theta+\theta^2+(1+\frac{\gamma}{x_i})^{-\zeta_1}\big(\theta-\theta^2-\theta^2(1+\frac{\gamma}{y_i})^{-2\zeta_2}\big)}. \tag{23}$$

3.3 Gumbel's BE Copula

Utilizing Equation (13), the conditional density function of Y given $X = x$ is defined as:

$$f(y|X=x) = \frac{\gamma\zeta_2}{y^2}(1+\frac{\gamma}{y})^{-\zeta_2-1}\big(1-(1+\frac{\gamma}{x})^{-\zeta_1}\big)^{-\theta\ln(1-(1+\frac{\gamma}{y})^{-\zeta_2})}V, \tag{24}$$

where

$$V = \Big(1-\theta-\theta\ln(1-(1+\frac{\gamma}{y})^{-\zeta_2})+\theta\big[\theta\ln(1-(1+\frac{\gamma}{y})^{-\zeta_2})-1\big]\ln(1-(1+\frac{\gamma}{x})^{-\zeta_1})\Big).$$

and with the marginal parameters replaced by their corresponding estimates:

$$\begin{aligned}\frac{\partial\ln L(y|x)}{\partial\theta} &= -\sum_{i=1}^{n}\ln\big(1-(1+\frac{\gamma}{y_i})^{-\zeta_2}\big)\ln\big(1-(1+\frac{\gamma}{x_i})^{-\zeta_1}\big)\\ &+\sum_{i=1}^{n}\frac{-1-\ln(1-(1+\frac{\gamma}{y_i})^{-\zeta_2})+\ln(1-(1+\frac{\gamma}{x_i})^{-\zeta_1})\big[-1+2\theta\ln(1-(1+\frac{\gamma}{y_i})^{-\zeta_2})\big]}{V_i}.\end{aligned} \tag{25}$$

3.4 GH copula

The conditional density function of Y given $X = x$, as derived from Equation (16), is expressed as

$$\begin{aligned}f(y|X=x) &= -\zeta_1^{\theta-1}(1+\frac{\gamma}{x})^{\zeta_1}\ln(1+\frac{\gamma}{x})^{\theta-1}\frac{\gamma(1-\theta)\zeta_2}{y(y+\gamma)}Z^{\frac{1}{\theta}-2}\exp(-Z^{\frac{1}{\theta}})\\ &\times\ln(1+\frac{\gamma}{y})^{\theta-1}+\frac{\gamma\zeta_2^{\theta}\exp(-Z^{\frac{1}{\theta}})\ln(1+\frac{\gamma}{y})^{\theta-1}Z^{\frac{2}{\theta}-2}}{y(y+\gamma)},\end{aligned} \tag{26}$$

where

$$Z = \zeta_1^{\theta}\ln(1+\frac{\gamma}{x})^{\theta}+\zeta_2^{\theta}\ln(1+\frac{\gamma}{y})^{\theta}.$$

and with the marginal parameters replaced by their corresponding estimates

$$\begin{aligned}\frac{\partial\ln L(y|x)}{\partial\theta} &= -n\ln\zeta_1-\frac{n}{\theta}+\sum_{i=1}^{n}\ln\big(\ln(1+\frac{\gamma}{x_i})\big)+2\sum_{i=1}^{n}\ln\big(\ln(1+\frac{\gamma}{y_i})\big)\\ &+n\ln\zeta_2-2\sum_{i=1}^{n}Z_i^{\frac{1}{\theta}}\big(S_i-\frac{\ln Z_i}{\theta^2}\big)-\frac{3}{\theta^2}\sum_{i=1}^{n}\ln Z_i+(\frac{3}{\theta}-3)\sum_{i=1}^{n}S_i,\end{aligned} \tag{27}$$

where

$$\begin{aligned}S_i &= \Big[\zeta_1^{\theta}\ln(1+\frac{\gamma}{x_i})^{\theta}\ln\big(\ln(1+\frac{\gamma}{x_i})\big)+\zeta_1^{\theta}\ln\zeta_1\ln(1+\frac{\gamma}{x_i})^{\theta}\\ &+\zeta_2^{\theta}\ln(1+\frac{\gamma}{y_i})^{\theta}\ln\big(\ln(1+\frac{\gamma}{y_i})\big)+\zeta_2^{\theta}\ln\zeta_2\ln(1+\frac{\gamma}{y_i})^{\theta}\Big]/Z_i.\end{aligned}$$

3.5 Asymptotic Properties of the Likelihood-Based Estimators

Consider a bivariate random sample $(X_i, Y_i); i = 1, 2, \ldots, n$, drawn from the joint distribution of (X, Y). Let the parameter vector be denoted by $\eta = (\zeta_1, \zeta_2, \gamma, \theta)$, and its estimator by $\hat{\eta} = (\hat{\zeta}_1, \hat{\zeta}_2, \hat{\gamma}, \hat{\theta})$, which is known to be a consistent estimator of η (see [14]). To establish the asymptotic distribution, define $g = (\frac{\partial \ln L_1}{\partial \zeta_1}, \frac{\partial \ln L_1}{\partial \gamma}, \frac{\partial \ln L_2}{\partial \zeta_2}, \frac{\partial \ln L_2}{\partial \gamma}, \frac{\partial \ln L}{\partial \theta}) = (g_1, g_2, g_3, g_4, g_5)$, denote a row vector, such that

$$\begin{aligned} L_1 &= \frac{\gamma^n \zeta_1^n}{\prod_{i=1}^n x_i^2} \prod_{i=1}^n x_i^2 (1 + \frac{\gamma}{x_i})^{-\zeta_1 - 1}, \\ L_2 &= \frac{\gamma^n \zeta_2^n}{\prod_{i=1}^n y_i^2} \prod_{i=1}^n y_i^2 (1 + \frac{\gamma}{y_i})^{-\zeta_2 - 1}. \end{aligned}$$

L represents the conditional likelihood of Y_i^{rs} given X_i^{rs} and is influenced by the chosen copula function. We get

$$\begin{aligned} g_1 &= \frac{\partial \ln L_1}{\partial \zeta_1} = \frac{n}{\zeta_1} - \sum_{i=1}^n \ln(1 + \frac{\gamma}{x_i}), \\ g_2 &= \frac{\partial \ln L_1}{\partial \gamma} = \frac{n}{\gamma} - (\zeta_1 + 1) \sum_{i=1}^n \frac{1}{x_i + \gamma}, \\ g_3 &= \frac{\partial \ln L_2}{\partial \zeta_2} = \frac{n}{\zeta_2} - \sum_{i=1}^n \ln(1 + \frac{\gamma}{y_i}), \\ g_4 &= \frac{\partial \ln L_2}{\partial \gamma} = \frac{n}{\gamma} - (\zeta_2 + 1) \sum_{i=1}^n \frac{1}{y_i + \gamma}. \end{aligned}$$

and g_5 for FGM, AMH, Gumbel's BE and GH copulas are provided in (21), (23), (25) and (27) respectively.

Define $\hat{\eta} = (\hat{\zeta}_1, \hat{\zeta}_2, \hat{\gamma}, \hat{\theta})$ as the estimator derived from the two-stage estimation process and let $\underline{X}_n = (X_1, \ldots, X_n)$ and $\underline{Y}_n = (Y_1, \ldots, Y_n)$. The asymptotic distribution of $\sqrt{n}(\hat{\eta} - \eta)^T$ is asymptotically equivalent to that of $\Big[-E\big(\frac{\partial g^T(\underline{X}_n, \underline{Y}_n, \eta)}{\partial \eta}\big)\Big]^{-1} Z$, where $Z \sim N\big(0, cov(g(\underline{X}_n, \underline{Y}_n, \eta))\big)$, see [14]. The asymptotic variance-covariance matrix of $\sqrt{n}(\hat{\eta} - \eta)^T$, commonly referred to as the inverse Godambe information matrix, is discussed in Joe [14].

$$V = D_g^{-1} \, M_g \, (D_g^{-1})^T,$$

where

$$D_g = E\big[\frac{\partial g^T(\underline{X}_n, \underline{Y}_n, \eta)}{\partial \eta}\big] = E \begin{bmatrix} \frac{\partial g_1}{\partial \zeta_1} & \frac{\partial g_1}{\partial \zeta_2} & \frac{\partial g_1}{\partial \gamma} & \frac{\partial g_1}{\partial \theta} \\ \frac{\partial g_2}{\partial \zeta_1} & \frac{\partial g_2}{\partial \zeta_2} & \frac{\partial g_2}{\partial \gamma} & \frac{\partial g_2}{\partial \theta} \\ \frac{\partial g_3}{\partial \zeta_1} & \frac{\partial g_3}{\partial \zeta_2} & \frac{\partial g_3}{\partial \gamma} & \frac{\partial g_3}{\partial \theta} \\ \frac{\partial g_4}{\partial \zeta_1} & \frac{\partial g_4}{\partial \zeta_2} & \frac{\partial g_4}{\partial \gamma} & \frac{\partial g_4}{\partial \theta} \\ \frac{\partial g_5}{\partial \zeta_1} & \frac{\partial g_5}{\partial \zeta_2} & \frac{\partial g_5}{\partial \gamma} & \frac{\partial g_5}{\partial \theta} \end{bmatrix}, \tag{28}$$

and

$$M_g = E\big[g^T(\underline{X}_n, \underline{Y}_n, \eta) g(\underline{X}_n, \underline{Y}_n, \eta)\big] = E\begin{bmatrix} g_1^2 & g_1g_2 & g_1g_3 & g_1g_4 & g_1g_5 \\ g_2g_1 & g_2^2 & g_2g_3 & g_2g_4 & g_2g_5 \\ g_3g_1 & g_3g_2 & g_3^2 & g_3g_4 & g_3g_5 \\ g_4g_1 & g_4g_2 & g_4^2 & g_4^2 & g_4g_5 \\ g_5g_1 & g_5g_2 & g_5^2 & g_5g_4 & g_5^2 \end{bmatrix}, \tag{29}$$

Let $\frac{\partial g_1}{\partial \zeta_1} = -\frac{n}{\zeta_1^2}$, therefore $E\Big(\frac{\partial g_1}{\partial \zeta_1}\Big) = -\frac{n}{\zeta_1^2}$. Similarly, $E\Big(\frac{\partial g_1}{\partial \gamma}\Big) = \sum_{i=1}^n E\Big(\frac{1}{x_i+\gamma}\Big)$, $E\Big(\frac{\partial g_1}{\partial \zeta_2}\Big) = E\Big(\frac{\partial g_1}{\partial \theta}\Big) = 0$, $E\Big(\frac{\partial g_2}{\partial \zeta_1}\Big) = -\sum_{i=1}^n E\Big(\frac{1}{x_i+\gamma}\Big)$, $E\Big(\frac{\partial g_2}{\partial \gamma}\Big) = -\frac{n}{\gamma^2}$, $E\Big(\frac{\partial g_2}{\partial \zeta_2}\Big) = E\Big(\frac{\partial g_2}{\partial \theta}\Big) = 0$, $E\Big(\frac{\partial g_3}{\partial \zeta_2}\Big) = -\frac{n}{\zeta_2^2}$, $E\Big(\frac{\partial g_3}{\partial \gamma}\Big) = \sum_{i=1}^n E\Big(\frac{1}{y_i+\gamma}\Big)$, $E\Big(\frac{\partial g_3}{\partial \zeta_1}\Big) = E\Big(\frac{\partial g_3}{\partial \theta}\Big) = 0$, $E\Big(\frac{\partial g_4}{\partial \zeta_2}\Big) = -\sum_{i=1}^n E\Big(\frac{1}{y_i+\gamma}\Big)$, $E\Big(\frac{\partial g_4}{\partial \gamma}\Big) = -\frac{n}{\gamma^2}$, $E\Big(\frac{\partial g_4}{\partial \zeta_1}\Big) = E\Big(\frac{\partial g_3}{\partial \theta}\Big) = 0$.

Hence,

$$D_g = \begin{bmatrix} -\frac{n}{\zeta_1^2} & 0 & \frac{n}{\zeta_1+\gamma} & 0 \\ -\frac{n}{\zeta_1+\gamma} & 0 & -\frac{n}{\gamma^2} & 0 \\ 0 & -\frac{n}{\zeta_2^2} & \frac{n}{\zeta_2+\gamma} & 0 \\ 0 & -\frac{n}{\zeta_2+\gamma} & -\frac{n}{\gamma^2} & 0 \\ E\Big(\frac{\partial g_5}{\partial \zeta_1}\Big) & E\Big(\frac{\partial g_5}{\partial \zeta_2}\Big) & E\Big(\frac{\partial g_5}{\partial \gamma}\Big) & E\Big(\frac{\partial g_5}{\partial \theta}\Big) \end{bmatrix}, \tag{30}$$

An estimate of the asymptotic variance-covariance matrix is derived by substituting the estimates of ζ_1, ζ_2, γ and θ. Subsequently, the computation of the expected values within the matrices D_g and M_g is performed via Monte Carlo simulation techniques. Section 5 illustrates the application of the Godambe information criterion for a particular parameter configuration.

Additionally, for the copulas discussed in Section 2, R is a continuous function of η, represented as $R = h(\eta)$. The function h is also continuous in η, which means that $\hat{R} = h(\hat{\eta})$ serves as a consistent estimator for R.

Furthermore, since the function $h(.)$ has continuous partial derivatives of first order, the conditions for the Delta method are satisfied, allowing us to proceed with its application.

$$\sqrt{n}(\hat{R} - R) \xrightarrow{d} N\big(0, h'(\eta)\, V\, h'(\eta)^T\big),$$

where $h'(\eta) = \big(\frac{\partial h}{\partial \zeta_1}, \frac{\partial h}{\partial \zeta_2}, \frac{\partial h}{\partial \gamma}, \frac{\partial h}{\partial \theta}\big)$, and $\xrightarrow{d}$ denotes convergence in distribution.

4 Estimation using Blomqvist's beta

Blomqvist's beta is a rank-based dependence measure that evaluates the association between two variables around their medians, rather than means. This coefficient captures the probability that a randomly selected pair of observations will fall in the same quadrant with respect to the marginal medians. As a result, it is particularly robust to outliers and non-normal distributions. Initially introduced by Blomqvist [1] and later elaborated by Nelsen [18], this measure provides a simple yet informative summary of concordance near the center of the joint distribution. A closed-form expression exists for the population version of Blomqvist's beta, making it a useful tool in both theoretical and applied contexts.

$$\beta(C_\theta) = -1 + 4C_\theta(\frac{1}{2}, \frac{1}{2}). \tag{31}$$

Blomqvist's beta can be employed across a wide range of copula families, making it a flexible tool for assessing dependence structures. Notably, it often serves as a reliable approximation to other well-known rank-based measures such as Spearman's rho and Kendall's tau, particularly when the joint distribution is symmetric. As discussed in Nelsen [18], this measure retains its interpretability and robustness under various dependence settings. Given a random sample (X_i, Y_i), for $i = 1, 2, \ldots, n$, drawn from a continuous bivariate distribution, the empirical counterpart of Blomqvist's beta, denoted by β_n, is computed as follows:

$$\beta_n = \frac{n_1 - n_2}{n_1 + n_2}. \tag{32}$$

Let $\tilde{X}_n$ and $\tilde{Y}_n$ denote the sample medians of the random variables X and Y, respectively. The value n_1 represents the number of sample observations (X_i, Y_i) for which both components lie on the same side of their corresponding mediansi.e., X_i is greater than $\tilde{X}_n$ and Y_i is greater than $\tilde{Y}_n$, or X_i is less than $\tilde{X}_n$ and Y_i is less than $\tilde{Y}_n$. In contrast, n_2 counts the number of discordant observations, where one component lies above and the other below its sample medianspecifically, either X_i is greater than $\tilde{X}_n$ with Y_i is less than $\tilde{Y}_n$, or X_i is less than $\tilde{X}_n$ with Y_i is greater than $\tilde{Y}_n$. The asymptotic distribution of the empirical Blomqvist's beta, denoted by β_n, has been rigorously analyzed in both bivariate and multivariate contexts by Schmid and Schmidt [21]. The findings are as follows:

$$\sqrt{n}(\beta_n - \beta) \xrightarrow{d} N\big(0, \sigma^2_{\beta,C}\big) \qquad as \qquad n \to \infty$$

The asymptotic variance $\sigma^2_{\beta,C}$ is reported in Genest et al. [7] as

$$\begin{aligned} \sigma^2_{\beta,C} &= 16C(\frac{1}{2}, \frac{1}{2})\Big[1 - C(\frac{1}{2}, \frac{1}{2})\Big] + 4\Big[C_1(\frac{1}{2}, \frac{1}{2}) - C_2(\frac{1}{2}, \frac{1}{2})\Big]^2 \\ &\quad +16C(\frac{1}{2}, \frac{1}{2})\Big[- C_1(\frac{1}{2}, \frac{1}{2}) - C_2(\frac{1}{2}, \frac{1}{2}) + 2C_1(\frac{1}{2}, \frac{1}{2})C_2(\frac{1}{2}, \frac{1}{2})\Big], \end{aligned} \tag{33}$$

where $C_1(u, v) = \frac{\partial C(u,v)}{\partial u}$ and $C_2(u, v) = \frac{\partial C(u,v)}{\partial v}$ must be present throughout and continuous on the interval $[0, 1]^2$. Next, by solving the equation $\beta(C_\theta) = \beta_n$ for θ, we obtain the sample estimate $\theta_{\beta,n}$ of θ for a specific copula. For more information, refer to Genest et al. [7]. The estimator of θ can be represented as $\theta_{\beta,n} = g_\beta(\beta_n)$, where $\theta = g_\beta(\beta)$. It is important to note that the estimate of θ derived from

Blomqvist's beta does not rely on the estimates of the margin parameters. Additionally, if $g'_\beta(\beta)$ exists and is non-zero, the Delta method provides the asymptotic behavior of $\theta_{\beta,n}$ as follows

$$\sqrt{n}(\theta_{\beta,n} - \theta) \xrightarrow{d} N\big(0, \sigma^2_{\theta,C}\big) \qquad as \qquad n \to \infty$$

where $\sigma^2_{\theta,C} = \big[g'_\beta(\beta)\big]^2 \sigma^2_{\beta,C}$.

By utilizing the estimates $(\hat{\zeta}_1, \hat{\zeta}_2, \hat{\gamma}, \theta_{\beta,n})$ of the parameters $(\zeta_1, \zeta_2, \gamma, \theta)$, we obtain an estimate $\hat{R}_\beta$ of R.

It should be emphasized that, for the class of copulas introduced in Section 2, the function R depends continuously on the parameter vector $\eta = (\zeta_1, \zeta_2, \gamma, \theta)$. Moreover, the estimator $\hat{\eta}_\beta$ is a consistent estimator of η, as established in [7]. As a result of the continuous mapping theorem, the estimator $\hat{R}_\beta$ also converges in probability to R, implying its consistency. The upcoming subsections present further details on Blomqvist's beta and the asymptotic variance $\sigma^2_{\theta,C}$ corresponding to the copulas analyzed in Section 2.

4.1 FGM Copula

From (31), we derive that $\beta(C_\theta) = \theta/4$. Additionally, β is within the range $[-\frac{1}{4}, \frac{1}{4}]$. Thus, θ can be determined by inverting Blomqvist's beta, resulting in $\theta = 4\beta$. Genest et al. [7] provide a closed-form expression for the asymptotic variance of the estimator of Blomqvist's beta. This result can be readily validated using equation (33).

The explicit formula for the asymptotic variance of the estimator for β is provided in Genest et al. [7], and it can be readily confirmed using (33) that

$$\sigma^2_{\beta,C} = 1 - \frac{\theta^2}{16}.$$

For this copula, the function $g_\beta(\beta) = 4\beta$, which means $g'_\beta(\beta) = 4$. Consequently, the asymptotic variance of $\theta_{\beta,C}$ can be expressed as

$$\sigma^2_{\theta,C} = \big[g'_\beta(\beta)\big]^2 \sigma^2_{\beta,C} = 16 - \theta^2.$$

4.2 AMH Copula

From (31), we get $\beta(C_\theta) = \frac{\theta}{4-\theta}$. Moreover, $\beta \in [-\frac{1}{5}, \frac{1}{3}]$. Thus, θ can be determined by inverting Blomqvist's beta, resulting in $\theta = \frac{4\beta}{1+\beta}$. Schmid and Schmidt [21] derived an explicit expression for the asymptotic variance of β_n, which can be directly verified through equation (33).

$$\sigma^2_{\beta,C} = \frac{16(\theta^4 - 7\theta^3 + 36\theta^2 - 80\theta + 64)}{(4-\theta)^5}.$$

In the case of this particular copula, the transformation function associated with Blomqvist's beta is given by $g_\beta(\beta) = \frac{4\beta}{1+\beta}$, which implies that its derivative is $g'_\beta(\beta) = \frac{4}{(1+\beta)^2}$. Consequently, the asymptotic variance of $\theta_{\beta,n}$ can be expressed as

$$\sigma^2_{\theta,C} = \left[g'_\beta(\beta)\right]^2 \sigma^2_{\beta,C} = \frac{\theta^4 - 7\theta^3 + 36\theta^2 - 80\theta + 64}{4-\theta}.$$

4.3 Gumbel's BE Copula

From (31), we have $\beta(C_\theta) = e^{-\theta(\ln 2)^2} - 1$. Additionally, β falls within the range of $[-0.381497, 0]$. Consequently, θ can be calculated by inverting Blomqvist's beta, resulting in the formula $\theta = -\frac{\ln(\beta+1)}{(\ln 2)^2}$.

The asymptotic variance $\sigma^2_{\beta,C}$ of the estimator β_n can be efficiently obtained once the quantities $C_\theta(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}e^{-\theta(\ln 2)^2}$ and $C_1(\frac{1}{2}, \frac{1}{2}) = C_2(\frac{1}{2}, \frac{1}{2}) = 1 - \frac{1}{2}(1 + \theta \ln 2)e^{-\theta(\ln 2)^2}$ are determined. Thus, (33) produces

$$\sigma^2_{\beta,C} = 4e^{-\theta(\ln 2)^2}\Big[e^{-\theta(\ln 2)^2}(\frac{3}{4} + \theta \ln 2) + 2\Big(1 - \frac{1}{2}(1 + \theta \ln 2)e^{-\theta(\ln 2)^2}\Big)^2 - 1\Big].$$

By utilizing the function $g_\beta(\beta) = -\frac{\ln(\beta+1)}{(\ln 2)^2}$ and its derivative $g'_\beta(\beta) = -\frac{1}{(\ln 2)^2(\beta+1)}$, the asymptotic variance of $\theta_{\beta,n}$ can be expressed as

$$\begin{aligned}\sigma^2_{\theta,C} &= \left[g'_\beta(\beta)\right]^2 \sigma^2_{\beta,C} = \frac{4}{(\ln 2)^4 e^{-\theta(\ln 2)^2}} \times \\ &\Big[e^{\theta(\ln 2)^2}(\frac{3}{4} + \theta \ln 2) + 2\Big(1 - \frac{1}{2}(1 + \theta \ln 2)e^{-\theta(\ln 2)^2}\Big)^2 - 1\Big].\end{aligned}$$

4.4 GH Copula

From (31), we get $\beta(C_\theta) = 4e^{-2^{\frac{1}{\theta}} \ln 2} - 1$. Moreover, $\beta \in [0, 1)$. Thus, θ is derived by inverting Blomqvist's beta to obtain $\theta = \frac{\ln 2}{\ln\left[\ln(\frac{4}{\beta+1})/\ln 2\right]}$. Schmid and Schmidt [21] provided the explicit formula for the asymptotic variance of β_n, and it can also be easily confirmed using (33) that

$$\sigma^2_{\beta,C} = 8h_\theta\Big[1 - 2h_\theta + (2^{\frac{1}{\theta}+1}h_\theta - 1)^2\Big]$$

where $h_\theta = e^{-2^{\frac{1}{\theta}} \ln 2}$. By considering the function $g_\beta(\beta) = \frac{\ln 2}{\ln \frac{\ln \frac{4}{\beta+1}}{\ln 2}}$ and its derivative $g'_\beta(\beta) = \frac{\ln 2}{(\beta+1)\ln(\frac{4}{\beta+1})\left[\ln \frac{\ln \frac{4}{\beta+1}}{\ln 2}\right]^2}$, the asymptotic variance of θ_β can be expressed as

$$\sigma^2_{\theta,C} = \left[g'_\beta(\beta)\right]^2 \sigma^2_{\beta,C} = \frac{\big(\ln 2\big)^2\left[1 - 2h_\theta + (2^{\frac{1}{\theta}+1}h_\theta - 1)^2\right]}{2h_\theta\big(\ln h_\theta\big)^2\left[\ln(-\ln h_\theta) - \ln(\ln 2)\right]^2}.$$

Additionally, to derive the asymptotic distribution of the estimator $R_{\theta,n}$, we examine two scenarios: (i) when $(\zeta_1, \zeta_2, \delta)$ are known, and (ii) when $(\zeta_1, \zeta_2, \delta)$ are unknown. These scenarios will be addressed in the subsequent subsection.

4.5 Asymptotic Properties of the Estimator of R

Let $\{(X_i, Y_i), i = 1, 2, \ldots, n\}$ denote a bivariate random sample drawn from the joint distribution of (X, Y). Define $\eta = (\zeta_1, \zeta_1, \delta)$. The functions $(g, g_1, g_2, g_3, g_4$ are as specified in Section 3.5, and let $g_5 = \beta_n - \beta$.

Case I: when $(\zeta_1, \zeta_1, \delta)$ **are known**.
Assuming the parameters $(\zeta_1, \zeta_1, \delta)$ are known, the estimator R can be expressed as $R_{\theta,n} = h_\theta(\theta_n)$, which depends solely on the parameter θ. Accordingly, the Delta method characterizes the asymptotic distribution of

$$\sqrt{n}(R_{\theta,n} - R) \underset{\rightarrow}{d} N\big(0, \sigma^2_{R,C}\big) \qquad as \qquad n \to \infty$$

where $\sigma^2_{R,C} = \big[h'_\theta(\theta)\big]^2 \sigma^2_{\theta,C}$.

However, due to the absence of a closed-form expression for R, deriving an explicit form for the asymptotic variance $\sigma^2_{R,C}$ of $R_{\theta,n}$ is infeasible. Therefore, a bootstrap-based approach is employed to obtain a consistent estimate of $\sigma^2_{R,C}$.

Case II: when $(\zeta_1, \zeta_1, \delta)$ **and** θ **are unknown**.
Define $\eta = (\zeta_1, \zeta_1, \delta, \theta)$ and denote by $\hat{\eta} = (\hat{\zeta}_1, \hat{\zeta}_1, \hat{\delta}, \theta_{\beta,n})$ the parameter estimates obtained via the two-stage estimation procedure. As outlined in Section 3.5, the asymptotic variance-covariance matrix of $\hat{\eta}$ is given by the inverse Godambe information matrix:

$$V = D_g^{-1} M_g (D_g^{-1})^T$$

where D_g and M_g are as in (28) and (29) respectively.

$$\Big(\sqrt{n}(\hat{\zeta}_1 - \zeta_1), \sqrt{n}(\hat{\zeta}_2 - \zeta_2), \sqrt{n}(\hat{\gamma} - \gamma), \sqrt{n}(\hat{\beta}_n - \beta)\Big) \underset{\rightarrow}{d} N\big(0, V\big) \;\; as \;\; n \to \infty.$$

Furthermore, since the estimator for θ can be expressed as $\theta_{\beta,n} = g_\beta(\beta_n)$, and provided that the derivative $g'_\beta(\beta_n)$ exists and is nonzero, the asymptotic distribution of $\theta_{\beta,n}$ follows from the Delta method:

$$\Big(\sqrt{n}(\hat{\zeta}_1 - \zeta_1), \sqrt{n}(\hat{\zeta}_2 - \zeta_2), \sqrt{n}(\hat{\gamma} - \gamma), \sqrt{n}(\hat{\theta}_{\beta,n} - \theta)\Big) \underset{\rightarrow}{d} N\big(0, V_1\big) \;\; as \;\; n \to \infty,$$

where

$$V_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g'_\beta(\beta) \end{bmatrix} V \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g'_\beta(\beta) \end{bmatrix}^T .$$

Moreover, the estimator of R can be represented as $R_{\theta,n} = h(\hat{\zeta}_1, \hat{\zeta}_2, \hat{\gamma}, \theta_{\beta,n})$, and the function $h(.)$ is continuously differentiable with respect to all its arguments. By applying the Delta method once more, we obtain:

$$\sqrt{n}(R_n - R) \underset{\rightarrow}{d} N\big(0, V_2\big) \qquad as \qquad n \to \infty$$

where

$$V_2 = \left(\frac{\partial R}{\partial \zeta_1}, \frac{\partial R}{\partial \zeta_2}, \frac{\partial R}{\partial \gamma}, \frac{\partial R}{\partial \theta}\right) V_1 \left(\frac{\partial R}{\partial \zeta_1}, \frac{\partial R}{\partial \zeta_2}, \frac{\partial R}{\partial \gamma}, \frac{\partial R}{\partial \theta}\right)^T.$$

5 Simulation study

To assess the effectiveness of the proposed estimators, multiple synthetic data sets were generated under various parameter configurations. Specifically, combinations of marginal parameters $(\zeta_1, \zeta_2, \gamma)$ such as $(1, 2, 1)$, $(2, 1, 1)$, $(0.5, 1.5, 2)$, $(1.5, 0.5, 2)$, $(0.3, 0.7, 0.5)$, and $(0.7, 0.3, 0.5)$ were considered in conjunction with several values of the dependence parameter θ, tailored to each of the four copula models discussed earlier.

For each specific parameter setting, 100 independent data sets were simulated, each consisting of 50 observations. The data generation process begins by drawing a random sample $(x_1, x_2, \ldots, x_n)$ from the IL distribution with parameters (ζ_1, γ). Subsequently, the corresponding values y_i are generated from the conditional distribution of Y given $X = x_i$. Marginal parameters ζ_1, ζ_2 and γ are then estimated using the MLE.

Let $\hat{\theta}$ and $\hat{R}$ denote the estimates of the dependence parameter θ and the function R, respectively, obtained through the two-stage ML estimation approach. Alternatively, $\theta_{\beta,n}$ and $R_{\theta,n}$ represent the corresponding estimators derived from the method based on Blomqvist's beta.

In the two-stage ML framework, estimation begins with computing $\hat{\theta}$, followed by the evaluation of $\hat{R}$ using the full set of estimated parameters $\hat{\eta} = (\hat{\zeta}_1, \hat{\zeta}_2, \hat{\gamma}, \hat{\theta})$. Simulation outcomes based on this methodology are reported in Table 1 for the FGM and AMH copulas, in Table 2 for Gumbel's bivariate extreme value (BE) copula, and in Table 3 for the GH copula. Within each cell of Tables 1 to 3, the first row displays the estimated value of θ, while the second row reports the associated Mean Squared Error (MSE). These results indicate that the estimates of θ are generally close to their corresponding true values.

Figure 2 illustrates the behavior of $\hat{R}$ across varying values of θ for all four copula models, under the six parameter settings $(\zeta_1, \zeta_2, \gamma)$ described in Section 2.5. As evidenced by the plots, the trend of $\hat{R}$ with respect to θ closely mirrors that of the true function R. Tables 4, 5, and 6 present the numerical values of $\hat{R}$ and the corresponding MSEs for each scenario.

Example of Godambe information: In order to calculate the Godambe information matrix, we utilized Monte Carlo methods to obtain the expected values $E\left(\frac{\partial g_5}{\partial \zeta_1}\right)$, $E\left(\frac{\partial g_5}{\partial \zeta_2}\right)$, $E\left(\frac{\partial g_5}{\partial \gamma}\right)$ and $E\left(\frac{\partial g_5}{\partial \theta}\right)$ in D_g, as well as all expected values in M_g, which are defined by equations (30) and (29) respectively. For instance, we examine the parameters $(\zeta_1, \zeta_2, \gamma, \theta) = (1, 2, 1, 0.5)$ for the FGM, AMH, and Gumbel's BE copulas, and $(\zeta_1, \zeta_2, \gamma, \theta) = (1, 2, 1, 5)$ for the GH copula. The inverse Godambe information matrix V is determined accordingly.

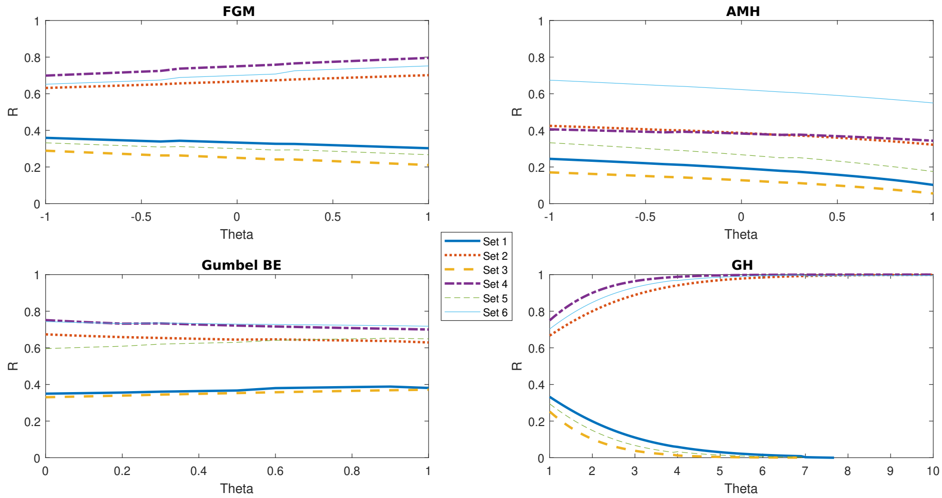

Figure 2: Variation in $\hat{R}$ against dependence parameter θ for the six sets.

FGM

$$\begin{bmatrix} 0.0011 & 0.0037 & -0.0041 & 0.1672 \\ 0.0037 & 0.0103 & -0.0172 & 0.9284 \\ -0.0041 & -0.0172 & 0.1928 & -56.1308 \\ 0.1672 & 0.9284 & -56.1308 & 462.7825 \end{bmatrix}$$

AMH

$$\begin{bmatrix} 0.0007 & 0.0014 & -0.0026 & -0.0359 \\ 0.0014 & 0.0362 & -0.0141 & -0.0158 \\ -0.0026 & -0.0141 & 0.0632 & 0.4830 \\ -0.0359 & -0.0158 & 0.4830 & 0.0129 \end{bmatrix}$$

Gumbel's BE

$$\begin{bmatrix} 0.0045 & 0.0023 & -0.0067 & -0.0461 \\ 0.0023 & 0.0171 & -0.0089 & -0.0103 \\ -0.0067 & -0.0089 & 0.7462 & 0.6631 \\ -0.0461 & -0.0103 & 0.6631 & 0.1267 \end{bmatrix}$$

GH

$$\begin{bmatrix} 0.0006 & 0.0009 & 0.0004 & -0.0507 \\ 0.0009 & 0.0224 & 0.0073 & -1.1165 \\ 0.0004 & 0.0073 & 0.1736 & -4.4812 \\ -0.0507 & -1.1165 & -4.4812 & 3.7612 \end{bmatrix}$$

In the second approach, the rank-based moment estimates of β_n are derived using (32). For large values of n, the estimator β_n follows a normal distribution with an asymptotic variance $\sigma_{\beta,C}$, as indicated in (33). The estimates for the dependence parameter, represented as $\theta_{\beta,n}$, are determined by solving the equation $\beta(C_\theta) = \beta_n$, by (31) and (32) to find θ. Furthermore, with the estimates $(\hat{\zeta}_1, \hat{\zeta}_2, \hat{\gamma}, \theta_{\beta,n})$, we can derive the estimate $R_{\theta,n}$.

The findings from the numerical investigation are presented in Tables 7, 8, 9 and 10. In each table, the estimate $\theta_{\beta,n}$ of θ, the MSE for $\theta_{\beta,n}$, and the 95% confidence intervals (CIs) for θ, calculated using a normal approximation, are displayed. Tables 7, 8, 9 and 10 reveal that the estimates $\theta_{\beta,n}$ for θ are generally near the actual value, with the exception of $\theta = 0.1$ for Gumbel's BE copula. Additionally, the MSE for $\theta_{\beta,n}$ is relatively high and tends to rise as θ increases in the case of the GH copula.

Figure 3 shows how $R_{\theta,n}$ changes in relation to θ for the four copulas discussed in Section 2.5. The figure reveals that the trend in the estimates $R_{\theta,n}$ of R as θ varies mirrors the trend in the actual R as θ changes. The values of $R_{\theta,n}$ and the MSEs for $R_{\theta,n}$ are presented in Tables 11, 12, and 13. It can be observed from Tables 11, 12, and 13 that the MSE values are relatively low; therefore, $R_{\theta,n}$ is considered a reliable estimator.

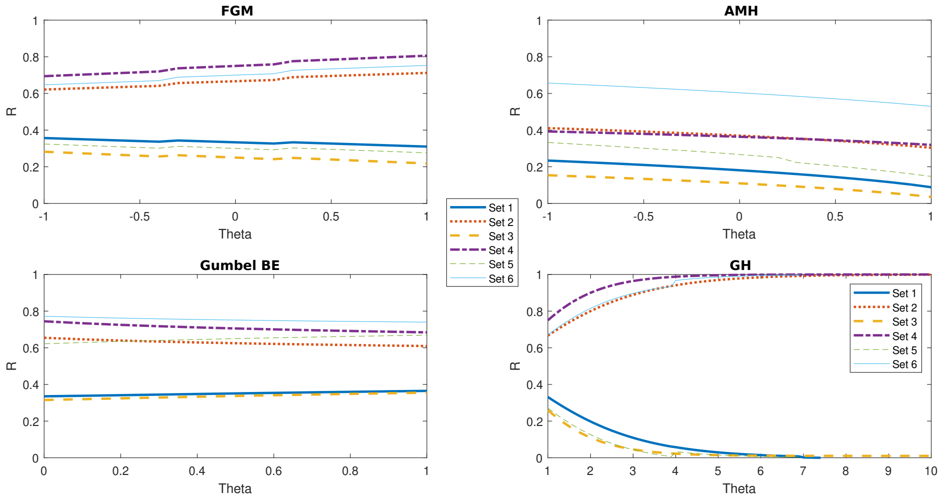

Figure 3: Variation in $R_{\theta,n}$ against dependence parameter θ for the six sets.

6 Application

We utilize the stress-strength data set reported in Tolba et al. [26] and reproduced in Table 14. We independently fitted the IL distribution to the datasets corresponding to variables X and Y. Table 15 presents the Kolmogorov-Smirnov (K-S) distance used to assess the goodness-of-fit, following the approach outlined in [2] along with the associated p-values for both variables, indicating that the IL distribution provides a good fit to each dataset.

Based on the data set, the dependence measures yield the following results: $\tau = 0.2359$, $\beta = 0.2937$, and the $\rho = 0.8412$. Based on the ranges of τ and β mentioned in the introduction, the AMH and GH copulas are suitable for modeling the dependency between the variables among the four copulas evaluated. The MLEs of the IL marginal parameters for variables X and Y are found to be $(\hat{\zeta}_1, \hat{\zeta}_2, \hat{\delta}) = (1.0077, 1.5314, 0.0915)$, respectively.

In Table 16, the estimates of the dependence parameter θ obtained via both the two-stage likelihood method $\hat{\theta}$ and the Blomqvist's beta approach $\theta_{\beta,n}$ are presented. The corresponding values of the target function R, namely $\hat{R}$ and $R_{\theta,n}$, are also shown for the AMH and GH copula families.

The AIC measure, is given by

$$\text{AIC} = -2 \ln L(\hat{\theta}, \hat{\theta} \text{ is the MLE}) + 2(\text{number of model parameters}).$$

For further details, the reader is referred to the Joe [14]. For the stress-strength test data, the AIC value under the independence assumption is 2355.7318. Therefore, dependent models provide a better fit. Moreover, it can be concluded that copula GH offers a better fit to the data compared to copula AMH.

7 Conclusions

This study systematically investigated the influence of copula selection on the reliability index R within a dependent stress-strength modeling framework, where both stress and strength variables follow IL distributions. Addressing the critical limitations of conventional independence assumptions, the research emphasizes the necessity of incorporating dependence structures to achieve more accurate reliability estimates in practical engineering contexts. Leveraging the flexibility of copula theory, the modeling framework decouples marginal behaviors from joint dependencies, allowing for the representation of a wide range of dependence patterns.

Closed-form expressions for R were derived for four widely used copula families, and the variation of reliability with respect to the dependence parameter θ was thoroughly examined. Two estimation techniques for $\theta$the conditional likelihood method and the method of moments based on Blomqvist's betawere employed, and their asymptotic distributions were studied through extensive Monte Carlo simulations. For copula cases where an analytical form of R was intractable, a Monte Carlo approach was utilized for numerical approximation.

Simulation results confirmed that the proposed methodology accurately captured the underlying behavior of reliability across various dependency levels and copula types. Notably, the GH copula demonstrated enhanced performance due to its

Table 1: Estimates $\hat{\theta}$ along with its MSE using FGM and AMH copula with the sample size $n = 50$.

Copula	Parameters		θ					
	$(\zeta_1, \zeta_2, \gamma)$		-0.9	-0.5	-0.1	0.1	0.5	0.9
FGM	(1,2,1)	$\hat{\theta}$	-0.8215	-0.4755	-0.0625	0.0924	0.4775	0.8377
		MSE	0.0915	0.1732	0.1992	0.2018	0.1592	0.0923
	(2,1,1)	$\hat{\theta}$	-0.8361	-0.4432	-0.1421	0.0641	0.4775	0.8251
		MSE	0.0672	0.1537	0.1983	0.1962	0.1655	0.0844
	(0.5,1.5,2)	$\hat{\theta}$	-0.8333	-0.4318	-0.1240	0.0631	0.5122	0.8385
		MSE	0.0613	0.1314	0.1845	0.1834	0.1357	0.07011
	(1.5,0.5,2)	$\hat{\theta}$	-0.8312	-0.4530	-0.1283	0.0682	0.4685	0.8286
		MSE	0.0806	0.1622	0.1953	0.1793	0.1492	0.0764
	(0.3,0.7,0.5)	$\hat{\theta}$	-0.8235	-0.4918	-0.1433	0.0793	0.4417	0.8179
		MSE	0.0705	0.1652	0.1811	0.1713	0.1592	0.0758
	(0.7,0.3,0.5)	$\hat{\theta}$	-0.8134	-0.4219	-0.0835	0.0851	0.5284	0.8512
		MSE	0.0915	0.1467	0.0925	0.1851	0.1697	0.0680
AMH	(1,2,1)	$\hat{\theta}$	-0.8431	-0.5207	-0.1063	0.0654	0.5064	0.8873
		MSE	0.0702	0.1176	0.1036	0.0872	0.0442	0.0525
	(2,1,1)	$\hat{\theta}$	-0.8514	-0.5384	-0.0710	0.0671	0.4618	0.8935
		MSE	0.0521	0.1223	0.0982	0.0781	0.0532	0.0161
	(0.5,1.5,2)	$\hat{\theta}$	-0.8602	-0.5216	-0.1346	0.1024	0.5183	0.8903
		MSE	0.0533	0.0982	0.0971	0.0591	0.0537	0.0623
	(1.5,0.5,2)	$\hat{\theta}$	-0.8681	-0.5820	-0.0670	0.1191	0.4833	0.8956
		MSE	0.0472	0.1253	0.09720	0.0787	0.0910	0.0063
	(0.3,0.7,0.5)	$\hat{\theta}$	-0.8552	-0.4662	-0.1113	0.0931	0.5043	0.8957
		MSE	0.0571	0.1087	0.0985	0.0980	0.0516	0.0527
	(0.7,0.3,0.5)	$\hat{\theta}$	-0.8492	-0.5177	-0.1232	0.0681	0.4786	0.8825
		MSE	0.0537	0.1204	0.1184	0.0884	0.0521	0.0520

wider support for strong positive dependence, reflected in the higher sensitivity of R to variations in θ relative to the FGM, AMH, and Gumbel's BE copulas. Additionally, the observed patterns of variation in estimated reliability with respect to θ were consistent with the corresponding theoretical trends across both estimation methods. Mean squared errors of the reliability estimates remained low for all copulas, and the conditional likelihood approach yielded estimates of θ closer to their true values compared to Blomqvist's beta.

Overall, this work provides valuable theoretical insights and practical guidance for selecting appropriate copula functions in stress-strength reliability analysis involving dependent variables. The proposed framework significantly enhances the robustness and precision of reliability assessments, offering important contributions to the modeling and design of complex engineering systems with inherent dependency structures.

Table 2: Estimates $\hat{\theta}$ along with its MSE using Gumbel's BE copula with the sample size $n = 50$.

Parameters		θ				
$(\zeta_1, \zeta_2, \gamma)$		0.1	0.3	0.5	0.7	0.9
(1,2,1)	$\hat{\theta}$	0.1321	0.3228	0.5321	0.6691	0.8815
	MSE	0.0271	0.0475	0.0426	0.0392	0.0235
(2,1,1)	$\hat{\theta}$	0.1282	0.3188	0.5347	0.7143	0.8717
	MSE	0.0295	0.0456	0.0526	0.0326	0.0286
(0.5,1.5,2)	$\hat{\theta}$	0.1266	0.3176	0.5268	0.6856	0.8773
	MSE	0.0275	0.0462	0.0492	0.0404	0.0358
(1.5,0.5,2)	$\hat{\theta}$	0.1217	0.3102	0.5317	0.7213	0.8954
	MSE	0.0214	0.0425	0.0437	0.0473	0.0195
(0.3,0.7,0.5)	$\hat{\theta}$	0.1253	0.3183	0.5271	0.6855	0.8736
	MSE	0.0217	0.0446	0.0482	0.0421	0.0255
(0.7,0.3,0.5)	$\hat{\theta}$	0.1118	0.3170	0.5172	0.7023	0.8712
	MSE	0.0178	0.0431	0.0301	0.0411	0.0402

Table 3: Estimates $\hat{\theta}$ along with its MSE using GH copula with the sample size $n = 50$.

Parameters		θ				
$(\zeta_1, \zeta_2, \gamma)$		2	4	6	8	10
(1,2,1)	$\hat{\theta}$	2.3782	4.2377	6.5317	7.7469	9.1036
	MSE	0.2513	0.4633	0.5782	0.7426	0.6662
(2,1,1)	$\hat{\theta}$	2.3305	4.2380	6.3604	7.8103	9.1176
	MSE	0.2276	0.4692	0.4218	0.7426	0.6381
(0.5,1.5,2)	$\hat{\theta}$	2.2476	4.1973	6.4521	7.8430	9.0817
	MSE	0.1875	0.3882	0.5174	0.5639	0.6857
(1.5,0.5,2)	$\hat{\theta}$	2.2054	4.1733	6.3207	7.8274	9.3126
	MSE	0.1640	0.3776	0.3176	0.6007	0.5127
(0.3,0.7,0.5)	$\hat{\theta}$	2.2533	4.1638	6.4122	7.7421	9.0646
	MSE	0.1976	0.3726	0.5108	0.7607	0.6833
(0.7,0.3,0.5)	$\hat{\theta}$	2.2190	4.1913	6.3712	7.8361	9.2033
	MSE	0.1833	0.3854	0.4283	0.6012	0.5985

Table 4: Estimates $\hat{R}$ along with its MSE using FGM and AMH copula with the sample size $n = 50$.

Copula	Parameters		θ					
	$(\zeta_1, \zeta_2, \gamma)$		-0.9	-0.5	-0.1	0.1	0.5	0.9
FGM	(1,2,1)	$\hat{R}$	0.3627	0.3510	0.3359	0.3313	0.3155	0.3025
		MSE	0.0027	0.0019	0.0023	0.0018	0.0029	0.0020
	(2,1,1)	$\hat{R}$	0.6356	0.6515	0.6621	0.6709	0.6825	0.6972
		MSE	0.0016	0.0022	0.0019	0.0018	0.0024	0.0021
	(0.5,1.5,2)	$\hat{R}$	0.2892	0.2721	0.2536	0.2450	0.2289	0.2106
		MSE	0.0019	0.0015	0.0021	0.0010	0.0008	0.0016
	(1.5,0.5,2)	$\hat{R}$	0.7123	0.7272	0.7451	0.7533	0.7709	0.7877
		MSE	0.0017	0.0029	0.0012	0.0018	0.0023	0.0014
	(0.3,0.7,0.5)	$\hat{R}$	0.3336	0.3194	0.3026	0.2954	0.2818	0.2664
		MSE	0.0017	0.0009	0.0021	0.0013	0.0011	0.0010
	(0.7,0.3,0.5)	$\hat{R}$	0.6645	0.6801	0.6948	0.7025	0.7178	0.7349
		MSE	0.0023	0.0015	0.0028	0.0021	0.0019	0.0012
AMH	(1,2,1)	$\hat{R}$	0.2421	0.2205	0.1985	0.1869	0.1541	0.1133
		MSE	0.0025	0.0018	0.0016	0.0012	0.0021	0.0017
	(2,1,1)	$\hat{R}$	0.4224	0.4086	0.3893	0.3815	0.3571	0.3277
		MSE	0.0015	0.0011	0.0017	0.0019	0.0010	0.0014
	(0.5,1.5,2)	$\hat{R}$	0.1683	0.1507	0.1321	0.1211	0.0976	0.0644
		MSE	0.0007	0.0016	0.0010	0.0012	0.0015	0.0008
	(1.5,0.5,2)	$\hat{R}$	0.4099	0.3972	0.3846	0.3798	0.3622	0.3453
		MSE	0.0013	0.0019	0.0024	0.0011	0.0019	0.0016
	(0.3,0.7,0.5)	$\hat{R}$	0.3276	0.3045	0.2753	0.2577	0.2238	0.1797
		MSE	0.0016	0.0026	0.0014	0.0018	0.0010	0.0006
	(0.7,0.3,0.5)	$\hat{R}$	0.6703	0.6519	0.6272	0.6172	0.5891	0.5565
		MSE	0.0013	0.0017	0.0025	0.0016	0.0009	0.0021

Table 5: Estimates $\hat{R}$ along with its MSE using Gumbel's BE copula with the sample size $n = 50$.

Parameters		θ				
$(\zeta_1, \zeta_2, \gamma)$		0.1	0.3	0.5	0.7	0.9
(1,2,1)	$\hat{R}$	0.3526	0.3615	0.3653	0.3734	0.3776
	MSE	0.0022	0.0017	0.0027	0.0013	0.0011
(2,1,1)	$\hat{R}$	0.6655	0.6531	0.6455	0.6366	0.6328
	MSE	0.0016	0.0014	0.0008	0.0023	0.0012
(0.5,1.5,2)	$\hat{R}$	0.3349	0.3437	0.3537	0.3625	0.3671
	MSE	0.0018	0.0012	0.0011	0.0008	0.0013
(1.5,0.5,2)	$\hat{R}$	0.7488	0.7344	0.7209	0.7106	0.7027
	MSE	0.0016	0.0015	0.0011	0.0014	0.0018
(0.3,0.7,0.5)	$\hat{R}$	0.6075	0.6213	0.6313	0.6394	0.6451
	MSE	0.0020	0.0014	0.0018	0.0019	0.0010
(0.7,0.3,0.5)	$\hat{R}$	0.7433	0.7344	0.7298	0.7230	0.7183
	MSE	0.0017	0.0025	0.0011	0.0020	0.0026

Table 6: Estimates $\hat{R}$ along with its MSE using GH copula with the sample size $n = 50$.

Parameters		θ				
$(\zeta_1, \zeta_2, \gamma)$		2	4	6	8	10
(1,2,1)	$\hat{R}$	0.2025	0.0604	0.0163	0.0033	9.7721×10^{-4}
	MSE	0.0027	0.0025	0.0017	0.0012	0.0006
(2,1,1)	$\hat{R}$	0.7986	0.9401	0.9834	0.9953	0.9981
	MSE	0.0019	0.0014	0.0016	0.0013	0.0011
(0.5,1.5,2)	$\hat{R}$	0.1012	0.0131	0.0022	1.4301×10^{-4}	1.66713×10^{-5}
	MSE	0.0015	0.0016	0.0010	0.0007	0.0004
(1.5,0.5,2)	$\hat{R}$	0.8989	0.9868	0.9989	0.9992	0.9989
	MSE	0.0013	0.0010	0.0012	0.0007	0.0005
(0.3,0.7,0.5)	$\hat{R}$	0.1545	0.0333	0.0057	0.0008	2.0825×10^{-4}
	MSE	0.0015	0.0018	0.0011	0.0009	0.0007
(0.7,0.3,0.5)	$\hat{R}$	0.8462	0.9665	0.9922	0.9966	0.9982
	MSE	0.0021	0.0017	0.0014	0.0019	0.0015

Table 7: Estimates $\theta_{\beta,n}$ along with its MSE and and 95% CIs using FGM copula with the sample size $n = 50$.

Parameters		θ					
$(\zeta_1, \zeta_2, \gamma)$		-0.9	-0.5	-0.1	0.1	0.5	0.9
(1,2,1)	$\theta_{\beta,n}$	-0.8623	-0.4918	-0.1130	0.1103	0.4892	0.9135
	MSE	0.1980	0.0208	0.1343	0.1059	0.1340	0.1103
	CI	(-0.9790,-0.7456)	(-0.5894,-0.3942)	(-0.2189,-0.0071)	(0.0117,0.2089)	(0.3900,0.5884])	(0.8143,1.0127)
(2,1,1)	$\theta_{\beta,n}$	-0.8713	-0.4739	-0.1280	0.0871	0.4931	0.9313
	MSE	0.1946	0.1870	0.1910	0.1235	0.0186	0.2218
	CI	(-0.9923,-0.7503)	(-0.5898,-0.3580)	(-0.2489,-0.0071)	(-0.0182,0.1924)	(0.4081,0.5782)	(0.8095,1.0532)
(0.5,1.5,2)	$\theta_{\beta,n}$	-0.8843	-0.4716	-0.1238	0.0735	0.4712	0.8901
	MSE	0.1271	0.1930	0.1802	0.1753	0.1941	0.0192
	CI	(-0.9941,-0.7745)	(-0.5956,-0.3477)	(-0.2342,-0.0135)	(-0.0375,0.1845)	(0.3471,0.5953)	(0.7965,0.9837)
(1.5,0.5,2)	$\theta_{\beta,n}$	-0.8491	-0.4912	-0.1351	0.1152	0.5135	0.9218
	MSE	0.2451	0.0219	0.1979	0.1266	0.1018	0.1412
	CI	(-0.9980,-0.7002)	(-0.5893,-0.3931)	(-0.2547,-0.0155)	(0.0060,0.2245)	(0.4139,0.6132)	(0.8209,1.0227)
(0.3,0.7,0.5)	$\theta_{\beta,n}$	-0.8912	-0.4831	-0.1106	0.0925	0.4813	0.9142
	MSE	0.0222	0.1248	0.0961	0.0179	0.1342	0.1115
	CI	(-0.9839,-0.7986)	(-0.5919,-0.3743)	(-0.2036,-0.0176)	(-0.0001,0.1851)	(0.3716,0.5911)	(0.8152,1.0133)
(0.7,0.3,0.5)	$\theta_{\beta,n}$	-0.8813	-0.4810	-0.1392	0.0846	0.4705	0.8812
	MSE	0.1350	0.1535	0.2118	0.1267	0.1985	0.1352
	CI	(-0.9906,-0.7720)	(-0.5937,-0.3683)	(-0.2611,-0.0173)	(-0.0160,0.1852)	(0.3488,0.5923)	(0.7718, 0.9906)

Table 8: Estimates $\theta_{\beta,n}$ along with its MSE and and 95% CIs using AMH copula with the sample size $n = 50$.

Parameters		θ					
$(\zeta_1, \zeta_2, \gamma)$		-0.9	-0.5	-0.1	0.1	0.5	0.9
(1,2,1)	$\theta_{\beta,n}$	-0.7913	-0.4980	-0.1318	0.0825	0.4811	0.9218
	MSE	0.2812	0.0026	0.1935	0.1341	0.1353	0.1412
	CI	(-0.9774,-0.6052)	(-0.5053,-0.4908)	(-0.2438,-0.0198)	(-0.0266,0.1916)	(0.3717,0.5906)	(0.8209,1.0228)
(2,1,1)	$\theta_{\beta,n}$	-0.8315	-0.4813	-0.1112	0.0769	0.5225	0.9130
	MSE	0.2493	0.1350	0.1333	0.2538	0.1865	0.1343
	CI	(-0.9876,-0.6755)	(-0.906,-0.3720)	(-0.2151,-0.0073)	(-0.1032,0.2570)	(0.4102,0.6349)	(0.8072,1.0189)
(0.5,1.5,2)	$\theta_{\beta,n}$	-0.9817	-0.4912	-0.1245	0.1208	0.4891	0.9310
	MSE	0.2559	0.0220	0.1852	0.2918	0.1340	0.1891
	CI	(-1.1743,-0.7892)	(-0.5894,-0.3930)	(-0.2352,-0.0138)	(-0.0723,0.3139)	(0.3899,0.5884)	(0.8198,1.0423)
(1.5,0.5,2)	$\theta_{\beta,n}$	-0.8711	-0.4776	-0.1210	0.0813	0.5092	0.8822
	MSE	0.1830	0.1783	0.1672	0.2512	0.0233	0.1347
	CI	(-0.9826,-0.7596)	(-0.5881,-0.3671)	(-0.2305,-0.0116)	(-0.0793,0.2419)	(0.4099,0.6085)	(0.7730,0.9915)
(0.3,0.7,0.5)	$\theta_{\beta,n}$	-0.9310	-0.4832	-0.1324	0.1242	0.4830	0.8910
	MSE	0.1841	0.1410	0.1959	0.2930	0.1413	0.0223
	CI	(-1.0441,-0.8179)	(-0.5842,-0.3823)	(-0.2510,-0.0139)	(0.0694,0.3178)	(0.3819,0.5841)	(0.7927,0.9893)
(0.7,0.3,0.5)	$\theta_{\beta,n}$	-0.8412	-0.4930	-0.1129	0.0913	0.4920	0.9225
	MSE	0.2461	0.0171	0.1013	0.2531	0.0176	0.1746
	CI	(-0.9835,-0.6989)	(-0.5855,-0.4005)	(-0.2182,-0.0076)	(-0.0716,0.2542)	(0.3993,0.5847)	(0.8125,1.0326)

Table 9: Estimates $\theta_{\beta,n}$ along with its MSE and and 95% CIs using Gumbel's BE copula with the sample size $n = 50$.

Parameters		θ				
$(\zeta_1, \zeta_2, \gamma)$		0.1	0.3	0.5	0.7	0.9
(1,2,1)	$\theta_{\beta,n}$	0.1064	0.3218	0.4862	0.6884	0.8821
	MSE	0.0151	0.1678	0.1025	0.1006	0.1349
	CI	(0.0147,0.1982)	(0.2122,0.4314)	(0.3806,0.5918)	(0.5848,0.7920)	(0.7728,0.9914)
(2,1,1)	$\theta_{\beta,n}$	0.1089	0.3201	0.5207	0.6879	0.8825
	MSE	0.0221	0.1658	0.1664	0.1012	0.1318
	CI	(0.0108,0.2071)	(0.2118,0.4285)	(0.4121,0.6293)	(0.5829,0.7930)	(0.7754,0.9896)
(0.5,1.5,2)	$\theta_{\beta,n}$	0.1151	0.3221	0.4828	0.6749	0.8836
	MSE	0.1225	0.1682	0.1315	0.1698	0.1305
	CI	(0.0101,0.2201)	(0.2123,0.4319)	(0.3762,0.5895)	(0.5640,0.7858)	(0.7781,0.9891)
(1.5,0.5,2)	$\theta_{\beta,n}$	0.1121	0.2795	0.5301	0.7175	0.8729
	MSE	0.1007	0.1652	0.1986	0.1336	0.1852
	CI	(0.0080,0.2162)	(0.1712,0.3879)	(0.4120,0.6482)	(0.6104,0.8242)	(0.7573,0.9885)
(0.3,0.7,0.5)	$\theta_{\beta,n}$	0.1095	0.3195	0.4819	0.6735	0.8775
	MSE	0.0190	0.1651	0.1372	0.1731	0.1685
	CI	(0.0161,0.2029)	(0.2121,0.4270)	(0.3749,0.5889)	(0.5620,0.7851)	(0.7676,0.9875)
(0.7,0.3,0.5)	$\theta_{\beta,n}$	0.1076	0.3285	0.5361	0.6709	0.8818
	MSE	0.0180	0.1782	0.2118	0.1806	0.0182
	CI	(0.0150,0.2002)	(0.2162,0.4408)	(0.4166,0.6557)	(0.5584,0.7835)	(0.7748,0.9889)

Table 10: Estimates $\theta_{\beta,n}$ along with its MSE and and 95% CIs using GH copula with the sample size $n = 50$.

Parameters		θ				
$(\zeta_1, \zeta_2, \gamma)$		2	4	6	8	10
(1,2,1)	$\theta_{\beta,n}$	2.0118	4.4118	6.5917	8.6418	9.1487
	MSE	0.1010	7.1208	9.3185	10.2610	11.5902
	CI	(1.9081,2.1155)	(3.9212,4.9024)	(5.4357,7.7477)	(7.3159,9.9677)	(7.7181,10.5794)
(2,1,1)	$\theta_{\beta,n}$	2.0161	4.4135	6.5908	8.6425	9.1459
	MSE	0.1304	7.1251	9.3151	10.2622	11.5944
	CI	(1.9107,2.1216)	(3.9219,4.9051)	(5.4366,7.7451)	(7.3160,9.9690)	(7.7137,10.5782)
(0.5,1.5,2)	$\theta_{\beta,n}$	2.0275	4.4070	6.6012	7.3618	9.1467
	MSE	0.1731	6.9702	9.3320	10.2572	11.5925
	CI	(1.9157,2.1393)	(3.9196,4.8945)	(5.4442,7.7583)	(6.0378,8.6859)	(7.7155,10.5779)
(1.5,0.5,2)	$\theta_{\beta,n}$	2.0360	4.4221	5.3967	7.3625	10.8447
	MSE	0.2111	7.2318	9.3375	10.2561	11.5852
	CI	(1.9162,2.1558)	(3.9220,4.9222)	(4.2386,6.5548)	(6.0388,8.6862)	(9.4176,12.2718)
(0.3,0.7,0.5)	$\theta_{\beta,n}$	2.0260	4.4121	6.5837	8.6431	9.1505
	MSE	0.1722	7.1238	9.3042	10.2635	11.5881
	CI	(1.9148,2.1372)	(3.9207,4.9035)	(5.4332,7.7342)	(7.3158,9.9705)	(7.7215,10.5796)
(0.7,0.3,0.5)	$\theta_{\beta,n}$	2.0280	4.4230	6.5930	7.3579	10.8559
	MSE	0.1736	7.2350	9.3220	10.2615	11.5961
	CI	(1.9158,2.1402)	(3.9225,4.9235)	(5.4366,7.7494)	(6.0317,8.6842)	(9.4223,12.2896)

Table 11: Estimates $\hat{R}_{\theta,n}$ along with its MSE using FGM and AMH copula with the sample size $n = 50$.

Copula	Parameters		θ					
	$(\zeta_1, \zeta_2, \gamma)$		-0.9	-0.5	-0.1	0.1	0.5	0.9
FGM	(1,2,1)	$\hat{R}$	0.3572	0.3518	0.3343	0.3322	0.3158	0.3042
		MSE	0.0021	0.0024	0.0043	0.0025	0.0023	0.0031
	(2,1,1)	$\hat{R}$	0.6352	0.6521	0.6625	0.6712	0.6842	0.6981
		MSE	0.0022	0.0037	0.0015	0.0023	0.0026	0.0039
	(0.5,1.5,2)	$\hat{R}$	0.2896	0.2728	0.2531	0.2433	0.2297	0.2097
		MSE	0.0027	0.0026	0.0025	0.0031	0.0017	0.0022
	(1.5,0.5,2)	$\hat{R}$	0.7134	0.7274	0.7446	0.7538	0.7711	0.7879
		MSE	0.0034	0.0025	0.0018	0.0014	0.0014	0.0012
	(0.3,0.7,0.5)	$\hat{R}$	0.3332	0.3196	0.3029	0.2957	0.2817	0.2668
		MSE	0.0019	0.0011	0.0017	0.0010	0.0009	0.0015
	(0.7,0.3,0.5)	$\hat{R}$	0.6648	0.6803	0.6955	0.7028	0.7182	0.7351
		MSE	0.0024	0.0012	0.0018	0.0019	0.0017	0.0015
AMH	(1,2,1)	$\hat{R}$	0.2423	0.2208	0.1982	0.1872	0.1547	0.1135
		MSE	0.0021	0.0016	0.0018	0.0015	0.0013	0.0014
	(2,1,1)	$\hat{R}$	0.4222	0.4088	0.3896	0.3818	0.3570	0.3275
		MSE	0.0017	0.0014	0.0026	0.0023	0.0012	0.0015
	(0.5,1.5,2)	$\hat{R}$	0.1689	0.1509	0.1318	0.1213	0.0979	0.0647
		MSE	0.0016	0.0014	0.0015	0.0009	0.0012	0.0011
	(1.5,0.5,2)	$\hat{R}$	0.4097	0.3967	0.3852	0.3804	0.3624	0.3451
		MSE	0.0010	0.0027	0.0019	0.0017	0.0021	0.0021
	(0.3,0.7,0.5)	$\hat{R}$	0.3272	0.3039	0.2762	0.2572	0.2245	0.1784
		MSE	0.0021	0.0016	0.0023	0.0025	0.0019	0.0013
	(0.7,0.3,0.5)	$\hat{R}$	0.6718	0.6511	0.6281	0.6175	0.5884	0.5566
		MSE	0.0016	0.0009	0.0011	0.0019	0.0014	0.0016

Table 12: Estimates $\hat{R}_{\theta,n}$ along with its MSE using Gumbel's BE copula with the sample size $n = 50$.

Parameters		θ				
$(\zeta_1, \zeta_2, \gamma)$		0.1	0.3	0.5	0.7	0.9
(1,2,1)	$\hat{R}$	0.3531	0.36215	0.3662	0.3739	0.3773
	MSE	0.0012	0.0023	0.0018	0.0009	0.0015
(2,1,1)	$\hat{R}$	0.6648	0.6527	0.6441	0.6389	0.6305
	MSE	0.0025	0.0019	0.0012	0.0013	0.0014
(0.5,1.5,2)	$\hat{R}$	0.3342	0.3455	0.3540	0.3594	0.3695
	MSE	0.0022	0.0011	0.0009	0.0011	0.0017
(1.5,0.5,2)	$\hat{R}$	0.7510	0.7341	0.7226	0.7110	0.7021
	MSE	0.0012	0.0013	0.0011	0.0009	0.0024
(0.3,0.7,0.5)	$\hat{R}$	0.6097	0.6194	0.6291	0.6392	0.6463
	MSE	0.0016	0.0011	0.0012	0.0013	0.0009
(0.7,0.3,0.5)	$\hat{R}$	0.7452	0.7366	0.7283	0.7254	0.7188
	MSE	0.0008	0.0007	0.0012	0.0013	0.0014

Table 13: Estimates $\hat{R}_{\theta,n}$ along with its MSE using GH copula with the sample size $n = 50$.

Parameters		θ				
$(\zeta_1, \zeta_2, \gamma)$		2	4	6	8	10
(1,2,1)	$\hat{R}$	0.2018	0.0580	0.0144	0.0048	0.0008
	MSE	0.0021	0.0008	0.0011	0.0010	0.0009
(2,1,1)	$\hat{R}$	0.8018	0.9422	0.9863	0.9978	0.9979
	MSE	0.0023	0.0012	0.0019	0.0017	0.0015
(0.5,1.5,2)	$\hat{R}$	0.0991	0.0117	0.0034	0.0081	0.0016
	MSE	0.0011	0.0009	0.0029	0.0015	0.0008
(1.5,0.5,2)	$\hat{R}$	0.9001	0.9895	0.9977	0.9971	0.9962
	MSE	0.0010	0.0014	0.0008	0.0017	0.0013
(0.3,0.7,0.5)	$\hat{R}$	0.1565	0.0316	0.0072	0.0026	1.0311×10^{-4}
	MSE	0.0019	0.0012	0.0008	0.0015	0.0009
(0.7,0.3,0.5)	$\hat{R}$	0.8438	0.9684	0.9941	0.9969	0.9976
	MSE	0.0027	0.0012	0.0007	0.0011	0.0021

Table 14: Real data (Strength X and stress Y).

X	0.853	0.759	0.866	0.809	0.717	0.544	0.492	0.403	0.344	0.213	0.116
	0.092	0.070	0.059	0.048	0.036	0.029	0.021	0.014	0.011	0.008	0.006
Y	0.853	0.759	0.874	0.800	0.716	0.557	0.503	0.399	0.334	0.207	0.118
	0.118	0.097	0.078	0.067	0.056	0.044	0.036	0.026	0.019	0.014	0.010

Table 15: Goodness-of-fit tests for fitting IL distribution to real data.

Variable	K-S statistic	P-value
X	0.1389	0.7665
Y	0.1408	0.7761

Table 16: Estimates $\hat{\theta}$, $\theta_{\beta,n}$, $\hat{R}$ and $R_{\theta,n}$, of θ and R respectively for FGM, AMH and GH copula for real data.

Copula	$\hat{\theta}$	$\hat{R}$	$\theta_{\beta,n}$	$R_{\theta,n}$	AIC
AMH	0.8190	0.9879	0.9735	0.8764	98.4182
GH	7.2842	0.9902	5.2218	0.9822	-175.3329

Bibliography

[1] N. Blomqvist, *On a measure of dependence between two random variables*, The Annals of Mathematical Statistics, 21:593–600, 1950.

[2] I. M. Chakravarti, R. G. Laha, and J. Roy, *Handbook of methods of applied statistics*, New York: Wiley, 1967.

[3] J. D. Church, B. Harris, *The Estimation of Reliability from Stress–Strength Relationships*, Technometrics, 12:49–54, 1970.

[4] F., Domma, S. Giordano, *A copula-based approach to account for dependence in stress–strength models*, Statistical Papers, 54:807–826, 2013.

[5] M. Émile, J. Gumbel, *Distributions des valeurs extremes en plusieurs dimensions*, In Annales de l'ISUP, 9:171–173, 1960.

[6] R. C. Gupta, M. E. Ghitany, and D. K. Al-Mutairi, *Estimation of Reliability from a Bivariate Log-Normal Data*, Journal of Statistical Computation and Simulation, 83:1068–1081, 2012.

[7] C. Genest, A. Carabarín-Aguirre, and F. Harvey, *Copula parameter estimation using Blomqvist's beta*, Journal de la Société Française de Statistique, 154:5–24, 2013.

[8] N. Hakamipour, *Gumbel copula-based reliability assessment to describe the dependence of the multicomponent stress–strength model for Pareto distribution*, Journal of Mathematics and Modeling in Finance, 4:1–17, 2024.

[9] N. Hakamipour, *Stress–strength reliability estimation of s-out-of-k multicomponent systems based on copula function for dependent strength elements under progressively censored sample*, International Journal of General Systems, 2024, https://doi.org/10.1080/03081079.2024.2405687.

[10] N. Hakamipour, *Copula for Dependent Competing Risks Model in Progressive-Stress Accelerated Life Tests*, Quality and Reliability Engineering International, 2025, https://doi.org/10.1002/qre.3811.

[11] P. Hougaard, *A class of multivanate failure time distributions*, Biometrika, 73:671–678, 1986.

[12] J. Hu, Y. Zhuang, and C. Goldiner, *Fixed-Accuracy Confidence Interval Estimation of $P(X < Y)$ under a Geometric–Exponential Model*, Japanese Journal of Statistics and Data Science, 4:1079–1104, 2021.

[13] Y. F. Jamilu, A. U. Adamu, Y. Aliyu, and I. I. Aliyu, *The Inverse Lomax distribution with applications*, In Royal Statistical Society Nigeria Local Group Annual Conference Proceedings, pages 105–112, 2025..

[14] H. Joe, *Multivariate Models and Dependence Concepts Monographs on Statistics and Applied Probability*, Chapmann & Hall London, 1997.

[15] M. Jovanović, V. Rajić, *Estimation of $P(X < Y)$ for Geometric-Exponential Model Based on Complete and Censored Samples*, Communications in Statistics – Simulation and Computation , 46:3050–3066, 2017.

[16] M. Jovanović, V. Rajić, *Estimation of $P(X < Y)$ for gamma exponential model*, CYUJOR , 24:283–291, 2014.

[17] S. Nadarajah, *Reliability for Some Bivariate Beta Distributions*, Mathematical Problems in Engineering, 2:101–111, 2005.

[18] R. B. Nelsen, *An Introduction to Copulas*, Springer, New York, 1999.

[19] D. Patil, U. V. Naik-Nimbalkar, and M. M. Kale, *Effect of Dependency on the Estimation of $P[Y < X]$ in Exponential Stress–strength Models*, Austrian Journal of Statistics, 51:10–34, 2022.

[20] D. Patil, U. V. Naik-Nimbalkar, and M. M. Kale, *Estimation of $P[Y < X]$ for dependence of stress–strength models with Weibull marginals*, Annals of Data Science 11:1303–1340, 2024.

[21] F. Schmid, R. Schmidt, *Nonparametric inference on multivariate versions of Blomqvist's beta and related measures of tail dependence*, Metrika, 66:323–354, 2007.

[22] S. Sengupta, *Unbiased Estimation of $P(X > Y)$ for Two-Parameter Exponential Populations Using Order Statistics*, Statistics, 45:179–188, 2011.

[23] A. Sharma, P. Kumar, *Estimation of Density and Reliability Function of Inverse Lomax Distribution under Type-II Censoring and the Sampling Scheme of Bartholomew*, International Journal of Applied Mathematics and Statistics, 59:129–150, 2020.

[24] S. K. Singh, U. Singh, and D. Kumar, *Bayes estimators of the reliability function and parameter of inverted exponential distribution using informative and non-informative priors*, Journal of Statistical computation and simulation, 83:2258–2269, 2013.

[25] A. SKLAR, *Fonctions De Repartitioné n Dimensions Et Leurs Marges*, Publications de l'Institut de Statistique de l'Université de Paris, 8:229–231, 1959..

[26] A. H. TOLBA, D. A. RAMADAN, E. M. ALMETWALLY, T. M. JAWA, AND N. SAYED-AHMED, *Statistical inference for stress–strength reliability using inverse Lomax lifetime distribution with mechanical engineering applications*, Thermal Science, 26(Spec. issue 1):303–326, 2022.

[27] A. WONG, *Interval Estimation of $P(Y < X)$ for Generalized Pareto Distribution*, Journal of Statistical Planning and Inference, 142:601–607, 2012.

[28] T. XAVIER, J. K. JOSE, *Stress–strength reliability estimation involving paired observation with ties using bivariate exponentiated half-logistic model*, Journal of Applied Statistics, 49:1049–1064, 2012.

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