Journal of Data Science and Modeling, Vol. 1, No. 2, 43-70, June 2023

Research Manuscript

# Risk Assessment of Interval-Valued Variables in Generalized Linear Models

Amir Masoud Malekfar[1], Farzad Eskandari*[2]

[1] PhD candidate, Department of Statistics, Faculty of Statistics, Mathematics and Computer, Allameh Tabataba'i University, Tehran, Iran.

[2] Professor of statistics, Department of Statistics, Faculty of Statistics, Mathematics and Computer, Allameh Tabataba'i University, Tehran, Iran.

Received: 02/11/2019 Accepted: 04/11/2021

**Abstract:** Interval-valued data are observed as ranges instead of single values and contain richer information than single-valued data. Meanwhile, interval-valued data are used for interval-valued characteristics, for instance, daily temperature, daily stock price, censoring times, grouped data, etc. Recent years have witnessed an increasing interest in interval-valued data analysis. Therefore, interval-valued variables have attracted unprecedented attention in the last decade. Recently, different linear regression approaches have been introduced to analyze interval-valued data. If distributions of response variables belong to the exponential family of distributions, the generalized linear models framework is used for modeling the relationships between interval-valued variables. An interval generalized linear model is proposed for the first time in this research. Then a suitable model is presented to estimate the parameters of the interval generalized linear model. The two models are provided based on interval arithmetic. The estimation procedure of the parameters of the suitable model is as the estimation procedure of the parameters of the interval generalized linear model. The least-squares (LS) estimation of the suitable model is developed according to a nice distance in the interval space. The LS estimation is resolved analytically through a constrained minimization problem. Then some desirable properties of the estimators are checked. Finally, both the theoretical and the empirical performance of the estimators are investigated.

**Keywords:** Interval-valued data; Interval arithmetic; Interval generalized linear model; Least-squares estimation.

**Mathematics Subject Classification (2010):** 62J12, 91G70.

*Corresponding Author: askandari@atu.ac.ir

# 1. Introduction

Over the past decade, interval-valued data have been considered as an attempt to overcome different sources of imprecision in generating and modeling single-valued (imprecise) data (see, for instance, Jahanshahloo et al., 2008). Based on the previous explanation, interval-valued data have been effectively used to represent imprecise data in recent industrial, economic and scientific studies. Censored and grouped data are also usually represented using intervals in general (see Calle and Gomez, 2001; Rivero and Valdes, 2008; Huber et al., 2009). In addition to these, intervals have to use as the values of interval-valued attributes, like ranges of a specific variable, fluctuations, physical measurements, subjective valuations, interval time sequences, and so on (see, for instance, Diamond, 1990; Gil et al., 2002, 2007).

Linear regression models have been most recently used for modeling the relationships between interval-valued random variables. Some of the regression analysis and modeling methods for interval-valued data have been deeply studied in the literature; see, for instance, Chesher and Irish (1987), Billard and Diday (2000), Gil et al. (2001), Manski and Tamer (2002), Hong and Tamer (2003), Neto et al. (2004), de Carvalho et al. (2004), Neto et al. (2005), Zhao et al. (2005), Billard (2006), Gonzalez-Rodriguez et al. (2007), Neto and de Carvalho (2008), Neto et al. (2009), Neto and de Carvalho (2010), and Wang et al. (2012). Bertrand and Goupil (2000) introduced the sample mean and the sample variance for interval-valued data. Also, Billard (2008, 2011) proposed the sample covariance between interval-valued data.

The relationships are modeled in the framework of generalized linear models if response variables have any statistical distributions belonging to the exponential family of distributions. For the first time in this paper, an interval generalized linear model is introduced and estimated based on interval arithmetic.

The rest of the paper consists of four sections. In Section 2, some introductory concepts are provided about the interval framework. In the rest of the section, the interval generalized linear model is introduced. Then a suitable model is defined to estimate the parameters of the interval generalized linear model, so that the estimation of the parameters of the suitable model is as the estimation of the parameters of the interval generalized linear model. In Section 3, the estimation of the parameters of the suitable model is obtained using a constrained minimization problem. In Section 4, some theoretical properties of the estimators are provided, and also the empirical performance of the obtained estimators is tested through some simulation studies. Finally, Section 5 states some conclusions.

# 2. Introductory concepts and an interval generalized linear model

## 2.1 Introductory concepts of the interval framework

The set of all closed real intervals is denoted as $\mathcal{I}(\mathbb{R})=\{[a,\ b] : a, b \in \mathbb{R},\ \mathrm{a}\leq\mathrm{b}\}$. Each interval $N\in\mathcal{I}(\mathbb{R})$ can be written as $N = [\inf\ N,\ \sup\ N]$, where $\inf\ N,\ \sup\ N \in \mathbb{R}$ and $\inf\ N\leq\sup\ N$. The interval $N$ can also be written as $N = [\mathrm{mid}\ N\pm\mathrm{spr}\ N] = \left[\frac{\inf\ N+\ \sup\ N}{2}\pm\frac{\sup\ N-\inf\ N}{2}\right]$, where mid $N \in \mathbb{R}$ and spr $N \in \mathbb{R}^+$ denote the *center* (or *midpoint* or *location*) and the *radius* (or *spread* or *imprecision*) of $N$, respectively. spr $N$ denotes the difference with a precise quantity of$\mathbb{R}$. In this paper, the (mid, spr)-parametrization for interval-valued data is used.

The natural interval arithmetic

$$Q_1+\lambda Q_2= [(\mathrm{mid}\ Q_1+\lambda\ \mathrm{mid}\ Q_2) \pm (\mathrm{spr}\ Q_1+|\lambda|\ \mathrm{spr}\ Q_2)]$$

for any $Q_1,\ Q_2 \in \mathcal{I}(\mathbb{R})$ and $\lambda \in \mathbb{R}$ is used to manage intervals. The space $(\mathcal{I}(\mathbb{R}), +,\ \bullet)$ is semi-linear due to the lack of symmetric element with respect to the addition; $Q_3 + (-1)\,Q_4$ can sometimes (not always) be equal to $Q_3 - Q_4$. For instance, $[0,\ 2] + (-1)\,[1,\ 2] = [-2,\ 1]$ and $[0,\ 2] - [1,\ 2] = [-1,\ 0]$ such that $[-2,\ 1] \neq [-1,\ 0]$. On the other hand, sometimes, Hukuhara difference $Q_5 = Q_3 - Q_4$ does not exists. For example, $Q_3 = [2,\ 5]$ and $Q_4 = [0,\ 6]$, so $Q_5 = Q_3 - Q_4 = [2, -1] \notin \mathcal{I}(\mathbb{R})$. Thus, statistical techniques in this space must always be developed by guaranteeing the coherency of the results with the semi-linear structure of $(\mathcal{I}(\mathbb{R}), +,\ \bullet)$.

The expression $\mathrm{E}([\mathrm{mid}\ B\pm\mathrm{spr}\ B])= [\mathrm{E}(\mathrm{mid}\ B) \pm \mathrm{E}(\mathrm{spr}\ B)]$ denotes the expected value of any random interval $B$ in terms of the Aumann expectation, whenever mid $B$, spr $B \in L^1$. Each $B \in \mathcal{I}(\mathbb{R})$ can be written based on the *canonical decomposition* of intervals as $B=$ mid $B\,[1 \pm 0] +B[0\pm1]$. Hence, we can apply mid $B$ and spr $B$ separately, but keeping the interval arithmetic connection.

In order to measure the distance between two intervals, an $L_2$-type metric has been exhaustively used and shown to be suitable on the space $\mathcal{I}(\mathbb{R})$. For every $A,\ Q \in \mathcal{I}(\mathbb{R})$, the *d-distance* is presented as

$$d\,(A,\ Q) = \sqrt{(\mathrm{mid}\ A - \mathrm{mid}\ Q)^2 + (\mathrm{spr}\ A - \mathrm{spr}\ Q)^2}. \tag{2.1}$$

Let $\mathcal{B}_d$ be the $\sigma$-field generated by the topology induced by $d$ on $\mathcal{I}(\mathbb{R})$. Let $(\Omega, \mathcal{A}, P)$ be a probability space. An interval-valued random variable $X$ is a $\mathcal{B}_d|\ \mathcal{A}$-measurable function $X :\ \Omega\longrightarrow\mathcal{I}(\mathbb{R})$. Equivalently, $\inf X, \sup X,\ \mathrm{mid}\ X, \mathrm{spr}\ X :$

$\Omega \to \mathbb{R}$, being real-valued random variables and spr $X \geq 0$ almost surely with respect to probability $P$. The interval-valued variable $X$ can not be a single-valued (or a real-valued) variable with respect to the non-degenerated variable spr $X$. The variance of $X$ with respect to $E(X) = [E(\text{mid } X) \pm E(\text{spr } X)]$ in the metric space $(\mathcal{I}(\mathbb{R}), d)$ is expressed as $\sigma^2(X) = E\left(d^2(X, E(X))\right) = \sigma^2(\text{mid } X) + \sigma^2(\text{spr } X)$, whenever $0 < \sigma^2(\text{mid } X), \sigma^2(\text{spr } X) < \infty$. The expressions $\sigma^2\left(X^M\right) = \sigma^2(\text{mid } X[1 \pm 0]) = \sigma^2(\text{mid } X)$ and $\sigma^2\left(X^S\right) = \sigma^2(\text{spr } X[0 \pm 1]) = \sigma^2([-\text{spr } X, \text{ spr } X]) = \sigma^2(\text{spr } X)$ can be easily proven. Thus, $\sigma(X, Y)$ is often defined as the corresponding $d$-covariance in $\mathbb{R}^2$ through the (mid, spr)-parametrization of the intervals, leading to the expression

$$\sigma(X, Y) = E[(\text{mid } X - E(\text{mid } X))(\text{mid } Y - E(\text{mid } Y))] \tag{2.2}$$

$$+ E[(\text{spr } X - E(\text{spr } X))(\text{spr } Y - E(\text{spr } Y))] = \sigma(\text{mid } X, \text{ mid } Y) + \sigma(\text{spr } X, \text{spr } Y), \tag{2.3}$$

whenever $\sigma(\text{mid } X, \text{ mid } Y), \sigma(\text{spr } X, \text{ spr } Y) < \infty$.

## 2.2 An interval generalized linear model for interval-valued data

The ordinary linear regression model uses linearity to describe the relationship between the mean of the response variable and a set of explanatory variables, with inference assuming that the response distribution is normal. Generalized linear models extend standard linear regression models to encompass non-normal response distributions and possibly nonlinear functions of the mean. The choice of distribution for a response variable determines the relation between the variance and the mean of the response variable, since the relation is known for many distributions. We now introduce an interval generalized linear model, for the first time in this section. The model is characterized as follows:

**(i)** We suppose that the observed realization $y_{ji}$ of the independent random response variable $Y_j$, $j = 1, 2$ and $i = 1, \ldots, n$, comes from a distribution that belongs to the exponential family of distributions with the density function

$$\pi(y_{ji}; \theta_j, \phi_j) = \exp\left[\psi_j(\phi_j)\{y_{ji}\theta_j - b_j(\theta_j) + K_j(y_{ji})\} + \omega_j(\phi_j, y_{ji})\right],$$

where $\psi_j(\phi_j) > 0$ so that $\phi_j$ is constant (see Nelder and Wedderburn, 1972). The parameter $\phi_j$, $j = 1, 2$, can be used as a nuisance parameter such as the variance $\sigma^2$ of a normal distribution. Hence, we will have $E(Y_1) = b\prime_1(\theta_1) = \mu$, $Var(Y_1) = \frac{1}{\psi_1(\phi_1)}\left(\frac{\partial \mu}{\partial \theta_1}\right) = \frac{b\prime\prime_1(\theta_1)}{\psi_1(\phi_1)} = v$,

**(ii)** For $i = 1, \ldots, n$, mid $x_i$ and spr $x_i$ are defined as real-valued random independent observations of mid $X$ and spr $X$, respectively, where mid $X$ and spr $X$ are real-valued independent variables. For $i = 1, \ldots, n$, mid $x_i\,[1 \pm 0]$ is defined as $x_i^M$, which is an observation of the variable $X^M = \text{mid } X\,[1 \pm 0]$. We can prove $x_i^S = -x_i^S$, $i = 1, \ldots, n$, where $x_i^S = [-\text{spr } x_i,\ \text{spr } x_i]$ is an interval-valued random independent observation of the interval variable $X^S = [-\text{spr } X,\ \text{spr } X]$. Also, $x_i = \text{mid } x_i\,[1 \pm 0] + \text{spr } x_i\,[0 \pm 1]$, $i = 1, \ldots, n$, is an interval-valued random independent observation of the interval-valued independent variable $X = \text{mid } X\,[1 \pm 0] + \text{spr } X\,[0 \pm 1]$, whenever $0 < \sigma^2(\text{mid } X),\ \ \sigma^2(\text{spr } X),\ \ \sigma^2(X) < \infty$. For $i = 1, \ldots, n$, the predicted part of the interval generalized linear model is presented based on the canonical decomposition as follows:

$$\begin{aligned} b_i &= \beta_0 \text{mid } x_i\,[1 \pm 0] + \beta_1 \text{spr } x_i\,[0 \pm 1] + \beta_2\,[1 \pm 0] + \beta_3\,[0 \pm 1] && (2.4)\\ &= \beta_0 x_i^M + \beta_1 x_i^S + [\beta_2 - \beta_3,\ \beta_2 + \beta_3], && (2.5) \end{aligned}$$

where $\beta_0 \in \mathbb{R}$ and $\beta_1 \geq 0$ are the model coefficients, and also $\beta_2 \in \mathbb{R}$ and $\beta_3 \geq 0$ are intercepts. Given $x_i^S = -x_i^S$ and $\beta_3\,[0 \pm 1] = -\beta_3\,[0 \pm 1]$, the predicted part can be written as

$$b_i = \beta_0 x_i^M + \beta_1 x_i^S + \beta_2\,[1 \pm 0] + \beta_3\,[0 \pm 1] = \beta_0 x_i^M + (-\beta_1)\, x_i^S + \beta_2\,[1 \pm 0] + (-\beta_3)\,[0 \pm 1],$$

for all $i$.

**(iii)** In this research, $z^{-1}(z(\mu)) = \mu$ and $h^{-1}(h(\varpi)) = \varpi$, where $z(.)$ and $h(.)$ are two monotone and differentiable functions, and are called link functions. The choice of link function is made based on the type of data. The link functions are called two canonical links so that $z(\mu) = \theta_1$ and $h(\varpi) = \theta_2$. In this paper, $z(\mu) = \theta_1 = \eta$ and $|h(\varpi)| = |\theta_2| = o$. The interval generalized linear model is proposed as follows:

$$\eta\,[1 \pm 0] + o\,[0 \pm 1] = \eta_i\,[1 \pm 0] + o_i\,[0 \pm 1] = \beta_0 x_i^M + \beta_1 x_i^S + [\beta_2 - \beta_3,\ \beta_2 + \beta_3],\ \ i = 1, \ldots, n,$$

where $\eta_i = \eta$, $o_i = o$, and $o\,[0 \pm 1] = -o\,[0 \pm 1]$. In this paper, the interval generalized linear model is called model SIM. To be more precise, model SIM can be written as the sum of the separate models

$$\eta = \eta_i = \beta_0 \text{ mid } x_i + \beta_2,$$

and

$$o = o_i = \beta_1 \text{ spr } x_i + \beta_3, \qquad i = 1, \ldots, n.$$

Given that $x_i^S = -x_i^S$, $o_i\,[0 \pm 1] = -o_i\,[0 \pm 1]$, and $\beta_3\,[0 \pm 1] = -\beta_3\,[0 \pm 1]$, for all $i$,

$$\begin{aligned}[\eta_i - o_i, \eta_i + o_i] &= \eta_i\,[1 \pm 0] + o_i\,[0 \pm 1] = \eta_i\,[1 \pm 0] + (-o_i)\,[0 \pm 1] \\ &= \beta_0 x_i^M + \beta_1 x_i^S + [\beta_2 - \beta_3, \beta_2 + \beta_3] \\ &= \beta_0 x_i^M + (-\beta_1)\ x_i^S + \beta_2\,[1 \pm 0] + (-\beta_3)\,[0 \pm 1]\end{aligned}$$

is proven, and the existence of the non-negative estimates of the parameters $\beta_1$ and $\beta_3$ of model SIM is guaranteed.

## 2.3 Model SIGL

If we use $\widehat{\eta}_i = z(\widehat{\mu}_i) = \widehat{\eta} = z(\widehat{\mu})$ and $\hat{o}_i = |h(\widehat{\varpi}_i)| = \hat{o} = |h(\widehat{\varpi})|$ instead of $\eta_i$ and $o_i$, respectively, in model SIM defined in Section 2.2, we will have $a_i = \text{mid } a_i\,[1 \pm 0] + \text{spr } a_i\,[0 \pm 1] = \widehat{\eta}_i\,[1 \pm 0] + \hat{o}_i\,[0 \pm 1]$, for $i = 1, \ldots, n$. The real-valued independent variables mid $G$ and spr $G$ will be introduced by replacing the real-valued dependent variables $Y_1$ and $Y_2$ instead of each of $y_{1i}$'s and $y_{2i}$'s in the expressions $\widehat{\eta}_i = \widehat{\eta} = z(\overline{y}_1) = z\left(\frac{y_{11}+\cdots+y_{1n}}{n}\right)$ and $\hat{o}_i = \hat{o} = |h(\overline{y}_2)| = \left|h\left(\frac{y_{21}+\cdots+y_{2n}}{n}\right)\right|$, respectively. For instance, for independent Poisson observations, $\widehat{\eta}_i = \widehat{\eta} = \log_e(\widehat{\mu}) = \log_e(\overline{y}_1)$ and $\hat{o}_i = \hat{o} = |\log_e(\widehat{\varpi})| = |\log_e(\overline{y}_2)|$, $i = 1, \ldots, n$, so mid $G = \log_e(Y_1)$, $G^M = \log_e(Y_1)\,[1 \pm 0]$, spr $G = |\log_e(Y_2)|$, and $G^S = |\log_e(Y_2)|\,[0 \pm 1]$. Therefore, when the single observations $y_{1i}$ and $y_{2i}$ are used instead of the variables $Y_1$ and $Y_2$ in the expressions mid $G$ and spr $G$, then mid $g_i$ and spr $g_i$ will be produced as independent random observations of the variables mid $G$ and spr $G$, respectively, $i = 1, \ldots, n$. For instance, for independent Poisson observations, mid $g_i = \log_e(y_{1i})$, $g_i^M = \log_e(y_{1i})\,[1 \pm 0]$, spr $g_i = |\log_e(y_{2i})|$, and $g_i^S = |\log_e(y_{2i})|\,[0 \pm 1]$, whenever $y_{1i}, y_{2i} > 0$, $i = 1, \ldots, n$, are, respectively, observations of the variables mid $G = \log_e(Y_1)$, $G^M = \log_e(Y_1)\,[1 \pm 0]$, spr $G = |\log_e(Y_2)|$, and $G^S = |\log_e(Y_2)|\,[0 \pm 1]$. Hence, $g_i = \text{mid } g_i\,[1 \pm 0] + \text{spr } g_i\,[0 \pm 1]$, $i = 1, \ldots, n$, is defined as an interval-valued random independent observation of the interval-valued variable $G = \text{mid } G\,[1 \pm 0] + \text{spr } G\,[0 \pm 1]$, whenever $0 < \sigma^2(\text{mid } G), \sigma^2(\text{spr } G), \sigma^2(G) < \infty$.

To estimate the parameters of model SIM, we propose the following model:

$$g_i = \beta_0\ x_i^M + \beta_1\ x_i^S + \beta_2\,[1 \pm 0] + e_i, \qquad i = 1, \ldots, n, \tag{2.6}$$

where $e_i = \text{mid } e_i\,[1 \pm 0] + \text{spr } e_i\,[0 \pm 1]$ is an unobserved interval-valued error of the interval-valued random error variable $\varepsilon = \text{mid } \varepsilon\,[1 \pm 0] + \text{spr } \varepsilon\,[0 \pm 1]$. On the other hand, $E\left(\varepsilon^M \mid X^M\right) =$, and also $E\left(\varepsilon^S \mid X^S\right) = [-\beta_3, \beta_3] \in \mathcal{I}(\mathbb{R})$ with $\beta_3 \geq 0$. In model (2), the intercept $\beta_2$ is also associated with mid $G$.

**Remark 2.3.1.** Model (2) is used to estimate the parameters of model SIM proposed in Section 2.2. Hence, the estimation of the parameters of model (2) is provided as the estimation of the parameters of model SIM.

The linear function associated with the model provided in (2) is expressed based on the canonical decomposition as follows:

$$E\left(G^M \mid X^M\right) + E\left(G^S \mid X^S\right) = \beta_0\, X^M + \beta_1\, X^S + \beta_4$$

where the interval-valued independent parameter $\beta_4$ is defined as $\beta_4 = [\beta_2 - \beta_3,\ \beta_2 + \beta_3] \in \mathcal{I}(\mathbb{R})$. Based on the expression $x_i^S = -x_i^S$, we can write $g_i = \beta_0\, x_i^M + \beta_1\, x_i^S + \beta_2\, [1 \pm 0] + e_i = \beta_0\, x_i^M + (-\beta_1)\ x_i^S + \beta_2\, [1 \pm 0] + e_i$, for all $i = 1, \ldots, n$. Hence, the existence of the non-negative estimate of the parameter $\beta_1$ is guaranteed by the existence of a double model in all the cases. It is straightforward to show that the following linear relationships for the *mid* and *spr* components of the intervals $g_i$ and $x_i$ are transferred from (2):

$$\begin{aligned} &\text{mid } g_i = \beta_0 \text{ mid } x_i + \beta_2 + \text{mid } e_i, && \textit{and} \\ &\text{spr } g_i = \beta_1 \text{ spr } x_i + \text{spr } e_i, && i = 1, \ldots,\ n. \end{aligned} \tag{2.7}$$

In the second model given in (3), the relationship between the *spr* observations sometimes (not always) coincides with the model of spr $g_i$ on spr $x_i$, $i = 1,\ \ldots,\ n$. The linear model of mid $g_i$ on mid $x_i$, $i = 1,\ \ldots,\ n$, always coincides with the relationship of the *mid* observations in the first model given in (3).

The transpose of a matrix $\mathbf{A}$of order $p$, $q$ is expressed by the notation $(\mathbf{A}_{p\times q})\prime = \mathbf{A}\prime_{q\times p}$. Let our notation for the inverse of an $n \times n$ square matrix $\mathbf{A}_1$ be $\mathbf{A}_1^{-1}$. The weight matrices are defined as follows:

$$\mathbf{w^M}_{n\times n} = \operatorname{diag}\left(w^M,\ \cdots,\ w^M\right) \tag{2.8}$$

$$\mathbf{w^S}_{n\times n} = \operatorname{diag}\ \left(w^S,\ \cdots,\ w^S\right) \tag{2.9}$$

where $w^M$ and $w^S$ are weights associated with the real-valued independent random response variables $Y_1$ and $Y_2$, respectively, so that

$$w^M = \left(\left(\frac{\partial\mu}{\partial\theta_1}\right)^2 (v)^{-1}\right) = \psi_1(\phi_1)\left(\frac{\partial\mu}{\partial\theta_1}\right)$$

and $w^S = \left(\left(\frac{\partial\varpi}{\partial\theta_2}\right)^2 (\Psi)^{-1}\right) = \psi_2(\phi_2)\left(\frac{\partial\varpi}{\partial\theta_2}\right)$.

Meanwhile, the variance matrices are provided as $\mathbf{v}_{n\times n} = \text{diag}\,(v,\ \ldots,\ v)$, and $\mathbf{\Psi}_{n\times n} = \text{diag}\,(\Psi,\ \ldots,\ \Psi)$.

The vectors are introduced as follows:

$$(\textbf{mid g})_{n\times 1} = \begin{pmatrix} \text{mid } g_1 \\ \vdots \\ \text{mid } g_n \end{pmatrix} = \left(\mathbf{g^M}\right)_{n\times 1} = \begin{pmatrix} g_1^M \\ \vdots \\ g_n^M \end{pmatrix} = \begin{pmatrix} \text{mid } g_1\,[1\pm 0] \\ \vdots \\ \text{mid } g_n\,[1\pm 0] \end{pmatrix},\quad (\textbf{spr g})_{n\times 1} = \begin{pmatrix} \text{spr } g_1 \\ \vdots \\ \text{spr } g_n \end{pmatrix} \tag{2.10}$$

$$\left(\mathbf{g^S}\right)_{n\times 1} = \begin{pmatrix} {g_1}^S \\ \vdots \\ {g_n}^S \end{pmatrix} = \begin{pmatrix} \text{spr } g_1\,[0\pm 1] \\ \vdots \\ \text{spr } g_n\,[0\pm 1] \end{pmatrix} = \begin{pmatrix} [-\text{spr } g_1,\ \text{spr } g_1] \\ \vdots \\ [-\text{spr } g_n,\ \text{spr } g_n] \end{pmatrix} \tag{2.11}$$

$$\mathbf{x^M}_{n\times 1} = \begin{pmatrix} x_1^M \\ \vdots \\ x_n^M \end{pmatrix} = \begin{pmatrix} \text{mid } x_1\,[1\pm 0] \\ \vdots \\ \text{mid } x_n\,[1\pm 0] \end{pmatrix} = (\textbf{mid x})_{n\times 1} = \begin{pmatrix} \text{mid } x_1 \\ \vdots \\ \text{mid } x_n \end{pmatrix}, (\textbf{spr x})_{n\times 1} = \begin{pmatrix} \text{spr } x_1 \\ \vdots \\ \text{spr } x_n \end{pmatrix} \tag{2.12}$$

$$\left(\mathbf{x^S}\right)_{n\times 1} = \begin{pmatrix} x_1^S \\ \vdots \\ x_n^S \end{pmatrix} = \begin{pmatrix} \text{spr } x_1\,[0\pm 1] \\ \vdots \\ \text{spr } x_n\,[0\pm 1] \end{pmatrix} = \begin{pmatrix} [-\text{spr } x_1, \text{spr } x_1] \\ \vdots \\ [-\text{spr } x_n, \text{spr } x_n] \end{pmatrix} \tag{2.13}$$

$$\mathbf{g}_{n\times 1} = \begin{pmatrix} g_1^M + g_1^S \\ \vdots \\ g_n^M + g_n^S \end{pmatrix} = \begin{pmatrix} \text{mid } g_1\,[1\pm 0] + \text{spr } g_1\,[0\pm 1] \\ \vdots \\ \text{mid } g_n\,[1\pm 0] + \text{spr } g_n\,[0\pm 1] \end{pmatrix} = \mathbf{g^M} + \mathbf{g^S} \tag{2.14}$$

$$\mathbf{x}_{n\times 1} = \begin{pmatrix} x_1^M + x_1^S \\ \vdots \\ x_n^M + x_n^S \end{pmatrix} = \begin{pmatrix} \text{mid } x_1\,[1\pm 0] + \text{spr } x_1\,[0\pm 1] \\ \vdots \\ \text{mid } x_n\,[1\pm 0] + \text{spr } x_n\,[0\pm 1] \end{pmatrix} = \mathbf{x^M} + \mathbf{x^S} \tag{2.15}$$

$$\begin{aligned}(\mathbf{e})_{n\times 1} &= \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} e_1^M + e_1^S \\ \vdots \\ e_n^M + e_n^S \end{pmatrix} = \begin{pmatrix} \text{mid } e_1\,[1\pm 0] + \text{spr } e_1\,[0\pm 1] \\ \vdots \\ \text{mid } e_n\,[1\pm 0] + \text{spr } e_n\,[0\pm 1] \end{pmatrix} = \begin{pmatrix} e_1^M \\ \vdots \\ e_n^M \end{pmatrix} + \begin{pmatrix} e_1^S \\ \vdots \\ e_n^S \end{pmatrix} \\ &= [1\pm 0]\,(\textbf{mid e})_{n\times 1} + [0\pm 1]\,(\textbf{spr e})_{n\times 1} = \left(\mathbf{e^M}\right)_{n\times 1} + \left(\mathbf{e^S}\right)_{n\times 1}\end{aligned} \tag{2.16}$$

where the vectors **spr e** , $\mathbf{e^M}$, **mid e**, and $\mathbf{e^S}$ are introduced as follows

$$(\textbf{spr e})_{n\times 1} = \begin{pmatrix} \text{spr } e_1 \\ \vdots \\ \text{spr } e_n \end{pmatrix}, \left(\mathbf{e^M}\right)_{n\times 1} = \begin{pmatrix} e_1^M \\ \vdots \\ e_n^M \end{pmatrix} = (\textbf{mid e})_{n\times 1} = \begin{pmatrix} \text{mid } e_1 \\ \vdots \\ \text{mid } e_n \end{pmatrix}$$

$$\text{and} \left(\mathbf{e^S}\right)_{n\times 1} = \begin{pmatrix} \text{spr } e_1\,[0\pm 1] \\ \vdots \\ \text{spr } e_n\,[0\pm 1] \end{pmatrix} = \begin{pmatrix} e_1^S \\ \vdots \\ e_n^S \end{pmatrix} \tag{2.17}$$

The model (2) can be expressed in matrix form as follows:

$$\mathbf{g}{=}\beta_0\ \mathbf{x}^{\mathbf{M}}{+}\beta_1\ \mathbf{x}^{\mathbf{S}}{+}\beta_2[1\pm 0]\ \mathbf{1}_{\boldsymbol{n}}{+}\mathbf{e}, \tag{2.18}$$

where $\mathbf{1}_{\boldsymbol{n}}$ is an $n\times 1$ vector of ones, and $\mathbf{g}$, $\mathbf{x}^{\mathbf{M}}$, $\mathbf{x}^{\mathbf{S}}$, and $\mathbf{e}$ are the $n\times 1$ vectors defined in (10), (8), (9), and (12), respectively. In this paper, the model (14) is called model SIGL, which is used to estimate the parameters of model SIM provided in Section 2.2. To be more precise, model SIGL can be expressed based on (3) as the sum of the following separate models:

$$\mathbf{mid\ g} = \beta_0\ \mathbf{mid\ x}{+}\beta_2\ \mathbf{1}_{\boldsymbol{n}}{+}\mathbf{mid\ e},$$

and

$$\mathbf{spr\ g} = \beta_1\ \mathbf{spr\ x}{+}\mathbf{spr\ e}, \tag{2.19}$$

where **mid g**, **spr g**, **mid x**, **spr x**, **mid e**, and **spr e** are the $n\times 1$ vectors given in (6), (7), (8), (9), (13), and (13), respectively.

# 3. Estimation of model SIGL

According to Remark 2.3.1, the estimation procedure of the parameters of model SIGL is applied as the estimation procedure of the parameters of model SIM given in Section 2.2.

In this paper, the notations $E(\text{mid } X)$, $E\left(X^M\right)$, $E(\text{mid } G)$, $E\left(G^M\right)$, $E(\text{spr } X)$, $E(\text{spr } G)$, $E\left(X^S\right)$, and $E\left(G^S\right)$ are used to denote the arithmetic means of the variables mid $X$, $X^M$, mid $G$, $G^M$, spr $X$, spr $G$, $X^S$, and $G^S$, respectively. The notations $\sigma^2(\text{mid } X)$, $\sigma^2\left(X^M\right)$, $\sigma^2(\text{mid } G)$, $\sigma^2\left(G^M\right)$, $\sigma^2(\text{spr } X)$, $\sigma^2(\text{spr } G)$, $\sigma^2\left(X^S\right)$, and $\sigma^2\left(G^S\right)$ are used to denote the arithmetic variances of the variables mid $X$, $X^M$, mid $G$, $G^M$, spr $X$, spr $G$, $X^S$, and $G^S$, respectively, defined in terms of the metric $d$ given in (1). We use the notation $\sigma(\text{spr } X, \text{spr } G)$ to denote the arithmetic covariance between the variables spr $X$ and spr $G$ (analogously $\sigma\left(X^S, G^S\right)$, $\sigma\left(X^M, G^M\right)$, and $\sigma(\text{mid } X, \text{mid } G)$).

Let the $i^{\text{th}}$ diagonal element of $\mathbf{w}^{\mathbf{M}}$ in (4) be the corresponding non-negative frequency weight of the $i^{\text{th}}$ element of the vectors $\mathbf{x}^{\mathbf{M}}$ (or **mid x**) and $\mathbf{g}^{\mathbf{M}}$ (or **mid g**) given in (8) and (6), respectively. The matrices $\mathbf{w}^{\mathbf{M}}$ in (4) and $\mathbf{w}^{\mathbf{S}}$ in (5) are two scalar matrices, so their diagonal elements do not effect the calculation of weighted arithmetic means and variances. We use the notations $\overline{X^M}$ (or $\overline{\text{mid } X}$) and $S^2\left(X^M\right)$ (or $S^2(\text{mid } X)$) to denote the arithmetic mean and variance of the sample elements given in $\mathbf{x}^{\mathbf{M}}$, respectively. Given $E\left(X^M\right)$ (or $E(\text{mid } X)$) and

$\sigma^2\left(X^M\right)$ (or $\sigma^2$ (mid $X$)), respectively, $\overline{X^M}$ and $S^2\left(X^M\right)$ are provided using the diagonal elements of $\mathbf{w^M}$ as follows:

$$\hat{E}\left(X^M\right)=\hat{E}\left(\text{mid }X\right)=\overline{X^M}=\overline{\text{mid }X}=\frac{\sum_{i=1}^n w^M\left(\text{mid }x_i\right)}{\sum_{i=1}^n w^M}=\frac{\sum_{i=1}^n x_i^M}{n}=\frac{x_1^M+\cdots+x_n^M}{n} \tag{3.20}$$

and

$$\begin{aligned}\widehat{\sigma}^2\left(X^M\right)&=\widehat{\sigma}^2\left(\text{mid }X\right)=S^2\left(X^M\right)=S^2\left(\text{mid }X\right)\\&=\frac{\sum_{i=1}^n w^M d^2\left(x_i^M,\ \overline{X^M}\right)}{\sum_{i=1}^n w^M}=\frac{\sum_{i=1}^n\left(x_i^M-\ \overline{X^M}\right)^2}{n}\end{aligned} \tag{3.21}$$

($\overline{X^S}$, $\overline{\text{spr }X}$, $\overline{G^M}=\overline{\text{mid }G}$, $\overline{G^S}$, $\overline{\text{spr }G}$, $S^2\left(X^S\right)=S^2\left(\text{spr }X\right)$, $S^2\left(G^M\right)=S^2\left(\text{mid }G\right)$, $S^2\left(G^S\right)=S^2\left(\text{spr }G\right)$ are expressed analogously based on the matrices and vectors given in (4) to (9)). The expression (17) is provided based on the metric (1). The notation $S\left(X^M,G^M\right)$ (or $S\left(\text{mid }X,\text{mid }G\right)$) is used to denote the covariance between the sample elements given in $\mathbf{x^M}$ (or **mid x**) and $\mathbf{g^M}$ (or **mid g**). Given $\sigma\left(X^M,G^M\right)$ (or $\sigma\left(\text{mid }X,\text{mid }G\right)$), $S\left(X^M,G^M\right)$ is presented using the diagonal elements of $\mathbf{w^M}$ as follows:

$$\begin{aligned}\widehat{\sigma}\left(X^M,G^M\right)&=\widehat{\sigma}\left(\text{mid }X,\text{mid }G\right)=S\left(X^M,G^M\right)=S\left(\text{mid }X,\text{mid }G\right)\\&=\frac{\sum_{i=1}^n w^M\left(x_i^M-\overline{X^M}\right)\left(g_i^M-\overline{G^M}\right)}{\sum_{i=1}^n w^M}=\frac{\sum_{i=1}^n\left(x_i^M-\overline{X^M}\right)\left(g_i^M-\overline{G^M}\right)}{n}\end{aligned} \tag{3.22}$$

($\widehat{\sigma}\left(X^S,G^S\right)=S\left(X^S,G^S\right)$ or $\widehat{\sigma}\left(\text{spr }X,\text{spr }G\right)=S\left(\text{spr }X,\text{spr }G\right)$ is written analogously based on the vectors given in (7) and (9) and the scalar matrix $\mathbf{w^S}$ given in (5)).

The matrices and vectors given in (4) to (11) are used to find the weighted LS (WLS) estimation of model SIGL. The objective function is provided with respect to (15) and the metric (1) as follows:

$$\begin{aligned}&\frac{\sum_{i=1}^n w^M d^2\left(\ g_i^M,\ ax_i^M+C^M\right)}{\sum_{i=1}^n w^M}+\frac{\sum_{i=1}^n w^S d^2\left(g_i^S,\ bx_i^S+\ C^S\right)}{\sum_{i=1}^n w^S}\\&\qquad=\frac{\sum_{i=1}^n d^2\left(\ g_i^M,\ ax_i^M+C^M\right)}{n}+\frac{\sum_{i=1}^n d^2\left(g_i^S,\ bx_i^S+\ C^S\right)}{n}.\end{aligned} \tag{3.23}$$

We see that the diagonal elements of the two matrices $\mathbf{w^M}$ and $\mathbf{w^S}$ have no effect on the calculation of the WLS estimation of model SIGL. The LS estimation of the parameters $(\beta_0,\ \beta_1,\ \beta_4)$ is obtained by minimizing the objective function (19) over $(a,\ b,\ C)$. The expression (19) can be written in terms of $(a,\ b,\ C)$ and the random intervals $g_i$'s and $x_i$'s with the finite second-order moments involved in model SIGL as follows:

$$\frac{\sum_{i=1}^{n}(\text{ mid } g_i - a \text{ mid } x_i - \text{ mid } C)^2}{n} + \frac{\sum_{i=1}^{n}(\text{ spr } g_i - b \text{ spr } x_i - \text{ spr } C)^2}{n} = f\left(a, \text{mid } C\right) + l(b, \text{ spr } C). \quad (3.24)$$

The of (20) over $(a,\ b,\ C)$ is calculated as follows:

$$\begin{cases} \frac{\partial f(a,\ \text{mid } C)}{\partial a} = -\frac{2}{n}\sum_{i=1}^{n}(\text{mid } g_i - a \text{ mid } x_i - \text{mid } C)(\text{mid } x_i) = 0 \\ \frac{\partial f(a,\ \text{mid } C)}{\partial \text{mid } C} = -\frac{2}{n}\sum_{i=1}^{n}(\text{mid } g_i - a \text{ mid } x_i - \text{mid } C) = 0 \\ \frac{\partial l(b,\ \text{spr } C)}{\partial b} = -\frac{2}{n}\sum_{i=1}^{n}(\text{spr } g_i - b \text{ spr } x_i - \text{spr } C)(\text{spr } x_i) = 0 \\ \frac{\partial l(b,\ \text{spr } C)}{\partial \text{spr } C} = -\frac{2}{n}\sum_{i=1}^{n}(\text{spr } g_i - b \text{ spr } x_i - \text{spr } C) = 0 \end{cases} \quad (3.25)$$

For the solution of the equations given in (21), using matrix algebra, the following matrix forms are provided:

$$\begin{cases} \left(\begin{pmatrix} 1 & \text{mid } x_1 \\ \vdots & \vdots \\ 1 & \text{mid } x_n \end{pmatrix}' \begin{pmatrix} 1 & \text{mid } x_1 \\ \vdots & \vdots \\ 1 & \text{mid } x_n \end{pmatrix}\right)^{-1} \begin{pmatrix} 1 & \text{mid } x_1 \\ \vdots & \vdots \\ 1 & \text{mid } x_n \end{pmatrix}' \begin{pmatrix} \text{mid } g_1 \\ \vdots \\ \text{mid } g_n \end{pmatrix} \\ \qquad = \binom{\text{mid } C}{a} = \begin{pmatrix} \text{mid } \widehat{\beta}_{4,LS} \\ \widehat{\beta}_{0,\ LS} \end{pmatrix} = \begin{pmatrix} \widehat{\beta}_{2,LS} \\ \widehat{\beta}_{0,\ LS} \end{pmatrix} \\ \left(\begin{pmatrix} 1 & \text{spr } x_1 \\ \vdots & \vdots \\ 1 & \text{spr } x_n \end{pmatrix}' \begin{pmatrix} 1 & \text{spr } x_1 \\ \vdots & \vdots \\ 1 & \text{spr } x_n \end{pmatrix}\right)^{-1} \begin{pmatrix} 1 & \text{spr } x_1 \\ \vdots & \vdots \\ 1 & \text{spr } x_n \end{pmatrix}' \begin{pmatrix} \text{spr } g_1 \\ \vdots \\ \text{spr } g_n \end{pmatrix} \\ \qquad = \binom{\text{spr } C}{b} = \begin{pmatrix} \text{spr } \widehat{\beta}_{4,LS} \\ \widehat{\beta}_{1,LS} \end{pmatrix} = \begin{pmatrix} \widehat{\beta}_{3,LS} \\ \widehat{\beta}_{1,LS} \end{pmatrix} \end{cases} \quad (3.26)$$

The LS estimators of the model SIGL (or model SIM) parameters are immediately obtained from (22) as follows:

$$\begin{cases} \widehat{\beta}_{2,LS} = \frac{\left(\sum_{i=1}^{n}(\text{mid } x_i)^2\right)\left(\sum_{i=1}^{n}(\text{mid } g_i)\right) - \left(\sum_{i=1}^{n}(\text{mid } x_i)\right)\left(\sum_{i=1}^{n}(\text{mid } x_i)(\text{mid } g_i)\right)}{n\left(\sum_{i=1}^{n}(\text{mid } x_i)^2\right) - \left(\sum_{i=1}^{n}(\text{mid } x_i)\right)^2} \\ \widehat{\beta}_{0,\ LS} = \frac{n\left(\sum_{i=1}^{n}(\text{mid } x_i)(\text{mid } g_i)\right) - \left(\sum_{i=1}^{n}(\text{mid } x_i)\right)\left(\sum_{i=1}^{n}(\text{mid } g_i)\right)}{n\left(\sum_{i=1}^{n}(\text{mid } x_i)^2\right) - \left(\sum_{i=1}^{n}(\text{mid } x_i)\right)^2} \\ \widehat{\beta}_{3,LS} = \frac{\left(\sum_{i=1}^{n}(\text{spr } x_i)^2\right)\left(\sum_{i=1}^{n}(\text{spr } g_i)\right) - \left(\sum_{i=1}^{n}(\text{spr } x_i)\right)\left(\sum_{i=1}^{n}(\text{spr } x_i)(\text{spr } g_i)\right)}{n\left(\sum_{i=1}^{n}(\text{spr } x_i)^2\right) - \left(\sum_{i=1}^{n}(\text{spr } x_i)\right)^2} \\ \widehat{\beta}_{1,LS} = \frac{n\left(\sum_{i=1}^{n}(\text{spr } x_i)(\text{spr } g_i)\right) - \left(\sum_{i=1}^{n}(\text{spr } x_i)\right)\left(\sum_{i=1}^{n}(\text{spr } g_i)\right)}{n\left(\sum_{i=1}^{n}(\text{spr } x_i)^2\right) - \left(\sum_{i=1}^{n}(\text{spr } x_i)\right)^2} \end{cases} \quad (3.27)$$

$$\begin{cases} \widehat{\beta}_{2,LS} = \frac{\left(\sum_{i=1}^{n}(\text{mid } x_i)^2\right)\left(\sum_{i=1}^{n}(\text{mid } g_i)\right)-\left(\sum_{i=1}^{n}(\text{mid } x_i)\right)\left(\sum_{i=1}^{n}(\text{mid } x_i)(\text{mid } g_i)\right)}{n\left(\sum_{i=1}^{n}(\text{mid } x_i)^2\right)-\left(\sum_{i=1}^{n}(\text{mid } x_i)\right)^2} \\ \widehat{\beta}_{0,\ LS} = \frac{n\left(\sum_{i=1}^{n}(\text{mid } x_i)(\text{mid } g_i)\right)-\left(\sum_{i=1}^{n}(\text{mid } x_i)\right)\left(\sum_{i=1}^{n}(\text{mid } g_i)\right)}{n\left(\sum_{i=1}^{n}(\text{mid } x_i)^2\right)-\left(\sum_{i=1}^{n}(\text{mid } x_i)\right)^2} \\ \widehat{\beta}_{3,LS} = \frac{\left(\sum_{i=1}^{n}(\text{spr } x_i)^2\right)\left(\sum_{i=1}^{n}(\text{spr } g_i)\right)-\left(\sum_{i=1}^{n}(\text{spr } x_i)\right)\left(\sum_{i=1}^{n}(\text{spr } x_i)(\text{spr } g_i)\right)}{n\left(\sum_{i=1}^{n}(\text{spr } x_i)^2\right)-\left(\sum_{i=1}^{n}(\text{spr } x_i)\right)^2} \\ \widehat{\beta}_{1,LS} = \frac{n\left(\sum_{i=1}^{n}(\text{spr } x_i)(\text{spr } g_i)\right)-\left(\sum_{i=1}^{n}(\text{spr } x_i)\right)\left(\sum_{i=1}^{n}(\text{spr } g_i)\right)}{n\left(\sum_{i=1}^{n}(\text{spr } x_i)^2\right)-\left(\sum_{i=1}^{n}(\text{spr } x_i)\right)^2} \end{cases} \tag{3.28}$$

The LS estimators $\widehat{\beta}_{0,\ LS}$, $\widehat{\beta}_{1,LS}$, $\widehat{\beta}_{2,LS}$, and $\widehat{\beta}_{3,LS}$ given in (23) can be presented simplicity as follows:

$$\begin{cases} \widehat{\beta}_{2,\ LS} = \overline{\text{mid } G} - \frac{\widehat{\sigma}(\text{mid } X,\ \text{mid } G)}{\widehat{\sigma}^2(\text{mid } X)}\ \overline{\text{mid } X} \\ \widehat{\beta}_{0,\ LS} = \frac{\widehat{\sigma}(\text{mid } X,\ \text{mid } G)}{\widehat{\sigma}^2(\text{mid } X)} \\ \widehat{\beta}_{3,LS} = \overline{\text{spr } G} - \frac{\widehat{\sigma}(\text{spr } X,\ \text{spr } G)}{\widehat{\sigma}^2(\text{spr } X)}\ \overline{\text{spr } X} \\ \widehat{\beta}_{1,LS} = \frac{\widehat{\sigma}(\text{spr } X,\ \text{spr } G)}{\widehat{\sigma}^2(\text{spr } X)} \end{cases} \tag{3.29}$$

The LS estimators given in (23) and (24) are proven in the Appendix.

Using the LS estimation of the first model given in (15), the minimization of the problem (20) can always be obtained over $a$ and mid $C$. The LS estimates can sometimes (not always) be suitable for estimating the model SIGL with respect to the semi-linearity of the interval space. Because, the minimization of the problem can sometimes (not always) be solved over $b$ and spr $C$ using the LS estimation of the second model given in (15). The LS estimator of $\beta_4$ is presented as follows:

$$\begin{aligned} \widehat{\beta}_{4,\ LS} &= C = C^M + C^S = \left[\widehat{\beta}_{2,LS} - \widehat{\beta}_{3,LS},\ \widehat{\beta}_{2,LS} + \widehat{\beta}_{3,LS}\right] \\ &= \left(\ \overline{G^M} + \overline{G^S}\ \right) - \left(\widehat{\beta}_{0,\ LS}\left(\overline{X^M}\right) + \widehat{\beta}_{1,LS}\left(\overline{X^S}\right)\right) \end{aligned} \tag{3.30}$$

$\widehat{\beta}_{4,\ LS}$ given in (25) is not sometimes well defined as a real interval because of the semi-linearity of $\mathcal{I}(\mathbb{R})$ (see Section 2.1).

In the minimization of the problem (19), the existence of a well-defined interval estimation of the parameter $\beta_4 = [\beta_2 - \beta_3,\ \beta_2 + \beta_3]$, and an appropriate estimation of the parameters $(\beta_0, \beta_1,\ \beta_2, \beta_3)$ have to be guaranteed via the existence of the Hukuhara differences (see Section 2.1). Hence, the existence of all the valid intervals $\left(g_i^M -\ \widehat{\beta}_0\ x_i^M\right) + \left(g_i^S -\ \widehat{\beta}_1\ x_i^S\right)$ with respect to the random observations given in (6) to (11) has to be verified by the estimators of the parameters of model SIGL (or model SIM). Given the necessity of the existence of the valid (or real or well-known) intervals (residuals), the minimization of the objective function (19) is written as follows:

$$\min\left(\frac{\sum_{i=1}^{n} d^2\left(\ g_i^M,\ a\ x_i^M+C^M\right)}{n}+\frac{\sum_{i=1}^{n} d^2\left(g_i^S,\ b\ x_i^S+\ C^S\right)}{n}\right)$$

subject to

$$\left(g_i^M+\ g_i^S\right)-\left(a\ x_i^M+b\ x_i^S\right)\ \ \text{exists, for all}\ \ i=1,\ \ldots,\ n \tag{3.31}$$

where $C$ and $b$ are unknown independent parameters (or quantities). The estimator of $\beta_4$ is obtained by solving (26) for $C$ as follows:

$$\widehat{\beta}_4=\left(\ \overline{G^M}+\overline{G^S}\ \right)-\left(\widehat{\beta}_0\left(\ \overline{X^M}\ \right)+\widehat{\beta}_1\left(\ \overline{X^S}\ \right)\right),$$

where the expressions $\widehat{\beta}_4$ and $\widehat{\beta}_{4,\ LS}$ are similar phrases. Based on the constraints of (26), in the expression $\widehat{\beta}_4$, the search for $\widehat{\beta}_0$ and $\widehat{\beta}_1$ has to be done in such a way that $\widehat{\beta}_4$ is a real interval. Problem (26) can be expressed considering the expression $\widehat{\beta}_4$ as

$$\begin{gathered}\min\ \tfrac{1}{n}\ \sum_{i=1}^{n}\ d^2\left(\ g_i^M-\ ax_i^M,\ \overline{G^M}-a\ \overline{X^M}\ \right)+\\ \min\ \tfrac{1}{n}\ \sum_{i=1}^{n}\ d^2\left(\ g_i^S-\ bx_i^S,\ \overline{G^S}-b\ \overline{X^S}\ \right)\end{gathered}$$

subject to

$$\left(g_i^M+\ g_i^S\right)-\left(a\ x_i^M+b\ x_i^S\right)\ \ \text{exists, for all}\ i=1,\ \ldots,\ n \tag{3.32}$$

The constraints can be written based on the quantity $b$, $g_i^S$'s, and $x_i^S$'s as follows:

$g_i^M+\ g_i^S-\left(a\ x_i^M+b\ x_i^S\right)$ exists

$$\Longleftrightarrow b\le\frac{\text{spr}\ g_i}{\text{spr}\ x_i}\ \ \text{for all}\ i=1,\ \ldots,\ n\ \ \text{such that spr}\ x_i\neq 0.$$

Therefore, the constraints of (26) are written as the set $(a,\ b)\in U=\mathbb{R}\times[0,\hat{u}_0]$, where

$$\hat{u}_0=\min\left\{\frac{\text{spr}\ g_i}{\text{spr}\ x_i}:\ \ \text{spr}\ x_i\neq 0\right\} \tag{3.33}$$

The re-solution of (27) over the set $U$ leads to obtain the estimators for the parameters of model SIGL (or model SIM) as follows:

$$\widehat{\beta}_0=\frac{\widehat{\sigma}\,(\text{mid}\ X,\ \text{mid}\ G)}{\widehat{\sigma}^2\,(\text{mid}\ X)}=\frac{\widehat{\sigma}\left(X^M,\ G^M\right)}{\widehat{\sigma}^2\left(X^M\right)} \tag{3.34}$$

$$\widehat{\beta}_1=\min\left\{\hat{u}_0,\ \max\left\{0,\frac{\widehat{\sigma}\,(\text{spr}\ X,\ \text{spr}\ G)}{\widehat{\sigma}^2\,(\text{spr}\ X)}\right\}\right\}\ =\min\left\{\hat{u}_0,\ \max\left\{0,\ \frac{\widehat{\sigma}\left(X^S,\ G^S\right)}{\widehat{\sigma}^2\left(X^S\right)}\right\}\right\} \tag{3.35}$$

$$\widehat{\beta}_2=\text{mid}\ \widehat{\beta}_4=\overline{\text{mid}\ G}-\ \frac{\widehat{\sigma}\,(\text{mid}\ X,\ \text{mid}\ G)}{\widehat{\sigma}^2\,(\text{mid}\ X)}\left(\ \overline{\text{mid}\ X}\ \right) \tag{3.36}$$

and

$$\widehat{\beta}_3 = \text{spr } \widehat{\beta}_4 \tag{3.37}$$

where

$$\begin{aligned}\widehat{\beta}_4 &= \left[\left(\ \overline{\text{mid } G} - \widehat{\beta}_0\ \overline{\text{mid } X}\ \right) \pm \left(\ \overline{\text{spr } G} - \widehat{\beta}_1\ \overline{\text{spr } X}\ \right)\right] \\ &= \left[\left(\ \overline{G^M} - \widehat{\beta}_0\ \overline{X^M}\ \right) \pm \left(\ \overline{G^S} - \widehat{\beta}_1\ \overline{X^S}\ \right)\right] \end{aligned} \tag{3.38}$$

The expressions (29) to (31) are provided with regard to $0<\widehat{\sigma}^2\,(\text{mid } X)<\infty$ (or equivalently $0 < \widehat{\sigma}^2\left(X^M\right) < \infty$) and $0 < \widehat{\sigma}^2\,(\text{spr } X) < \infty$ (or equivalently $0 < \widehat{\sigma}^2\left(X^S\right) < \infty$). $\widehat{\beta}_1$ in (30) and $\widehat{\beta}_3$ in (32) are obtained with respect to the set $U$. So, $\widehat{\beta}_3$, $\widehat{\beta}_4$ in (33), and $\widehat{\beta}_1$ are sometimes different from $\widehat{\beta}_{3,LS}$ in (24), $\widehat{\beta}_{4,LS}$ in (25), and $\widehat{\beta}_{1,LS}$ in (24) for the parameters $\beta_3$, $\beta_4$, and $\beta_1$ of model SIGL (or model SIM), respectively. However, $\widehat{\beta}_0$ and $\widehat{\beta}_2$ in (29) and (31), respectively, coincide with $\widehat{\beta}_{0,\ LS}$ in (24) and $\widehat{\beta}_{2,\ LS}$ in (24) for the first model given in (15) (or the model $\eta_i = \beta_0 \text{ mid } x_i+\beta_2$, $i = 1, \ldots, n$).

# 4. Statistical properties of the estimators

This section investigates two main properties of $\widehat{\beta}_0$, $\widehat{\beta}_1$, $\widehat{\beta}_2$, $\widehat{\beta}_3$, and $\widehat{\beta}_4$ of model SIGL (or model SIM).

## 4.1 Strong consistency results

The asymptotic unbiasedness and the strong consistency of the estimators given in (29) to (33) are demonstrated both theoretically and empirically in this section. The estimators given in (30), (32), and (33), which depend on min and max operators, are asymptotically unbiased estimators with respect to simulation results of the estimation process in Section 4.2. The estimators $\widehat{\beta}_0$ in (29) or $\widehat{\beta}_{0,\ LS}$ in (24) and $\widehat{\beta}_2$ in (31) or $\widehat{\beta}_{2,\ LS}$ in (24) are, respectively, unbiased estimators for the parameters $\beta_0$ and $\beta_2$ of the first model given in (15) (or the model $\eta_i = \beta_0 \text{ mid } x_i+\beta_2$, $i = 1, \ldots, n$), so that $E\left(\widehat{\beta}_0\right) = E\left(\widehat{\beta}_{0,\ LS}\right) = \beta_0$ and $E\left(\widehat{\beta}_2\right) = E\left(\widehat{\beta}_{2,LS}\right) = \beta_2$.

In this research, the number of simulations is denoted as NS. Let $\{x_{ik},\ g_{ik}\}_{i=1}^n$ be defined as the $k^{th}$, $k = 1,\ \ldots,\ NS$, simulated random sample of $n$ observations $\{x_i,\ g_i\}_{i=1}^n$ of $(X, G)$. So, $x_i$ and $g_i$ can be each of $x_{ik}$'s and $g_{ik}$'s, respectively,

$i = 1, \ldots, n$. The strong consistency of the estimators (29) to (33) of model SIGL (or model SIM) is demonstrated using the following result.

**Lemma 4.1.** *Let the elements given in the vectors* (11) *and* (10) *be two random samples from the interval-valued variables* $X$ *and* $G$, *respectively, verifying a model with the structure of model SIGL such that* $0 < \sigma^2(\text{spr } X), \sigma^2(\text{spr } G) < \infty$. *Then,*

$$P\left(\lim_{n\to\infty} \hat{u}_0(n) = u_0\right) = 1,$$

*where* $u_0 = \min_{\substack{i \leq k \\ NS \to \infty}} \left\{ \frac{\text{spr } g_{ik}}{\text{spr } x_{ik}} : \text{ spr } x_{ik} \neq 0 \right\}.$

It is clear that the expression $P\left(\lim_{n\to\infty} \hat{u}_0(n) = u_0\right) = 1$ is easily proven.

Theorem 4.1 presents the strong consistency of the estimators given in (29) to (33) based on the result obtained in Lemma 4.1.

**Theorem 4.1.** *Let the elements given in the vectors* (11) *and* (10) *be two random samples from the interval-valued variables* $X$ *and* $G$, *respectively, verifying a model SIGL, such that*

$0 < \sigma^2(\text{spr } X),\ \sigma^2(\text{spr } G),\ \sigma^2(\text{mid } X), \sigma^2(\text{mid } G) < \infty,$

*and* $\sigma(\text{spr } X, \text{spr } G),\ \sigma(\text{mid } X,\ \text{mid } G) < \infty.$

*The estimators* (29) *to* (33) *of model SIGL (or model SIM) are strongly consistent such that*

$\widehat{\beta}_0 \overset{n\to\infty}{\longrightarrow} \beta_0,\ \widehat{\beta}_1 \overset{n\to\infty}{\longrightarrow} \beta_1,\ \widehat{\beta}_2 \overset{n\to\infty}{\longrightarrow} \beta_2,\ \widehat{\beta}_3 \overset{n\to\infty}{\longrightarrow} \beta_3,$ *and* $\widehat{\beta}_4 \overset{n\to\infty}{\longrightarrow} \beta_4$ a.s.- [P].

The estimators given in (30), (32), and (33) are provided based on the constraints of (26). Therefore, the results provided in Lemma 4.1 and Theorem 4.1 can sometimes (not always) be obtained based on the estimators given in (24) and (25). The Appendix presents the proof of Theorem 4.1.

## 4.2 Simulation results of the estimation process

By considering the estimation procedure of the parameters of model SIGL as the estimation procedure of the parameters of model SIM, the empirical behavior of the estimation method presented in Section 3 is investigated through the Monte Carlo method when the parameters of model SIGL have known quantities. In this paper, the number of simulations or NS is 10000. The estimates $\widehat{\beta}_{0k}$, $\widehat{\beta}_{1k}$, $\widehat{\beta}_{2k}$ and $\widehat{\beta}_{3k}$ (with $\widehat{\beta}_{4k} = \left[\widehat{\beta}_{2k} - \widehat{\beta}_{3k},\ \widehat{\beta}_{2k} + \widehat{\beta}_{3k}\right]$), $k = 1, \ldots, 10000$, of the parameters $\beta_0$, $\beta_1$, $\beta_2$ and $\beta_3$ (with $\beta_4 = [\beta_2 - \beta_3,\ \beta_2 + \beta_3]$), respectively, are computed for $\{(x_{ik}, g_{ik})\}_{i=1}^{n}$, which is the $k^{th}$ simulated random sample of $n$, $n = 5,\ 50,\ 500$, observations $\{(x_i, g_i)\}_{i=1}^{n}$ of $(X, G)$.

Table 1: Experiential validation of the estimators with respect to the parameters of model SIM (or model SIGL).

| Model | $n$ | $\hat{E}\left(\widehat{\beta}_0\right)$ | $\widehat{MSE}\left(\widehat{\beta}_0\right)$ | $\hat{E}\left(\widehat{\beta}_1\right)$ | $\widehat{MSE}\left(\widehat{\beta}_1\right)$ | $\hat{E}\left(\widehat{\beta}_2\right)$ | $\widehat{MSE}\left(\widehat{\beta}_2\right)$ | $\hat{E}\left(\widehat{\beta}_3\right)$ | $\widehat{MSE}\left(\widehat{\beta}_3\right)$ |
|---|---|---|---|---|---|---|---|---|---|
| $SIGL_1$ | 5 | 3.002602 | 0.03343147 | 1.922313 | 0.12642170 | 4.998227 | 0.0580907 | 0.358235 | 0.01638056 |
| | 50 | 3.000827 | 0.00142998 | 1.987898 | 0.00103649 | 4.999478 | 0.0041022 | 0.357237 | 0.00132912 |
| | 500 | 2.999866 | 0.00013390 | 1.995829 | 0.00007203 | 5.000068 | 0.0003966 | 0.356818 | 0.00013728 |
| $SIGL_2$ | 5 | 4.999084 | 0.02304776 | 0.989408 | 0.00213924 | -3.00047 | 0.0064673 | 0.119120 | 0.00174733 |
| | 50 | 5.000142 | 0.00092868 | 0.997508 | 0.00003760 | -2.99998 | 0.0004601 | 0.119046 | 0.00015107 |
| | 500 | 4.999900 | 0.00008851 | 0.999132 | 0.00000281 | -3.00001 | 0.0000443 | 0.118962 | 0.00001461 |
| $SIGL_3$ | 5 | 1.002748 | 0.02184236 | 3.995723 | 0.00032320 | -0.02140 | 1.4287950 | 0.071117 | 0.00064383 |
| | 50 | 1.000307 | 0.00087458 | 3.998966 | 0.00000651 | -0.00205 | 0.0630781 | 0.071335 | 0.00005371 |
| | 500 | 0.999951 | 0.00008441 | 3.999608 | 0.00000057 | 0.00024 | 0.0060651 | 0.071363 | 0.00000528 |

The vector of 10000 values for each estimate will be defined as an empirical distribution of the corresponding estimator close to the theoretical one. Based on 10000 iterations, $\hat{E}\ (\widehat{\Upsilon}) = (\sum_{k=1}^{10000} \widehat{\Upsilon}_k)/10000$ and

$$\widehat{MSE}(\widehat{\Upsilon}) = \left( \sum_{k=1}^{10000} \left( \widehat{\Upsilon}_k - \Upsilon \right)^2 \right) /10000$$

are calculated, respectively, as the estimated mean value, and the estimated mean squared error of an estimator $\widehat{\Upsilon}$ from the empirical distribution, as empirical approximations of the theoretical ones.

Three models with the structure of model SIM are studied in this section. Different parameters are investigated in the models. To estimate the parameters of each model, a corresponding model SIGL is introduced. In the models with the structure of model SIGL, mid $x_i$'s, mid $g_i$'s, spr $x_i$'s, and spr $g_i$'s come from different distributions. On the other hand, for $i = 1, \ldots, n$, $\{\text{mid } g_{ik}\}_{k=1}^{10000}$, $\{\text{spr } g_{ik}\}_{k=1}^{10000}$, $\{\text{mid } x_{ik}\}_{k=1}^{10000}$, $\{\text{spr } x_{ik}\}_{k=1}^{10000}$, $\{\text{mid } e_{ik}\}_{k=1}^{10000}$, and $\{\text{spr } e_{ik}\}_{k=1}^{10000}$ are simulated random samples of mid $G$, spr $G$, mid $X$, spr $X$, mid $\varepsilon$, and spr $\varepsilon$, respectively. The models, taking into account that observed values of the spr variables have to be non-negative, will be employed.

1. **Model SIM$_\mathbf{1}$:** Let the independent observed responses $y_{1i}$'s and $y_{2i}$'s come from $Poisson(5)$, so $\eta_i = \log_e(5)$, $o_i = |\log_e(5)|$, and $w_i^M = w_i^S = 5$, $i = 1, \ldots, n$. Based on the delta method, mid $\varepsilon \sim N(0, \frac{1}{5})$ and spr $\varepsilon = |KK|$ such that $KK \sim N(0, \frac{1}{5})$, hence, $E\left(\varepsilon^S \mid X^S\right) = [-\beta_3, \beta_3] = [-0.357,\ 0.357]$ and $E\left(\varepsilon^M \mid X^M\right) = 0$. Assume that the independent observations mid $x_i$'s and spr $x_i$'s come from $N(0,)$ and $\chi_1^2$, respectively. We consider the model SIM$_1$ as

   $$\eta_i\,[1 \pm 0] + o_i\,[0 \pm 1] = \beta_0 \text{ mid } x_i\,[1 \pm 0] + \beta_1 \text{ spr } x_i\,[0 \pm 1] + \beta_2\,[1 \pm 0] + \beta_3\,[0 \pm 1]$$

   $i = 1, \ldots, n$, where $\beta_0 = 3$, $\beta_1 = 2$, $\beta_2 = 5$, and $\beta_3 = 0.35679$. To estimate the parameters of model SIM$_1$, let $g_i$ be a random interval determined by using the following model:

   $$g_i = 3x_i^M - 2x_i^S + 5\,[1 \pm 0] + e_i = 3x_i^M + 2x_i^S + 5\,[1 \pm 0] + e_i \qquad i = 1, \ldots, n$$

   The model SIGL$_1$ is as

   $$\mathbf{g} = 3\ \mathbf{x^M} - 2\ \mathbf{x^S} + 5\,[1 \pm 0]\ \mathbf{1}_n + \mathbf{e} = 3\ \mathbf{x^M} + 2\ \mathbf{x^S} + 5\,[1 \pm 0]\ \mathbf{1}_n + \mathbf{e} \qquad (4.39)$$

Table 2: Generated interval-valued data set.

| $g$ | $x$ | $g$ | $x$ | $g$ | $x$ |
|---|---|---|---|---|---|
| [0.9555, 5.5606] | [15, 35] | [1.9459, 3.3322] | [10, 20] | [1.9459, 4.7184] | [2, 26] |
| [1.7917, 3.9889] | [12, 30] | [0.6931, 4.8520] | [6, 30] | [1.3350, 4.5538] | [6, 24] |
| [1.6582, 4.4308] | [6, 22] | [1.5260, 4.7449] | [7, 27] | [0.3101, 5.1059] | [13, 33] |
| [1.4663, 5.0498] | [8, 32] | [0.4418, 4.8362] | [11, 19] | [1.0116, 5.1704] | [17, 25] |
| [1.2992, 4.8828] | [12, 22] | [0.8472, 4.4308] | [9, 23] | [0.9382, 5.3327] | [6, 26] |
| [1.8718, 3.2580] | [6, 26] | [1.5260, 4.7449] | [5, 29] | [0.8109, 4.9698] | [15, 31] |
| [0.9444, 4.8362] | [17, 29] | [1.6094, 4.8283] | [7, 27] | [1.9459, 3.3322] | [9, 29] |
| [0.1335, 4.0253] | [16, 32] | [1.1451, 5.0369] | [6, 18] | [1.4350, 4.6539] | [10, 22] |
| [0.7472, 5.1416] | [3, 21] | [1.1786, 3.9512] | [3, 23] | [1.0414, 4.6249] | [8, 26] |
| [0.8754, 4.0943] | [3, 21] | [0.8209, 5.6167] | [9, 25] | [1.7917, 3.1780] | [15, 31] |
| [1.3437, 4.9272] | [12, 28] | [1.0986, 4.6821] | [5, 11] | [1.1786, 3.9512] | [15, 35] |
| [0.9808, 4.5643] | [13, 39] | [1.6582, 4.4308] | [19, 29] | [1.1786, 3.9512] | [12, 24] |
| [1.3862, 4.9698] | [10, 22] | [1.7227, 4.9416] | [18, 26] | [0.3677, 4.7621] | [22, 32] |
| [1.5581, 4.3307] | [13, 19] | [1.5686, 4.7874] | [11, 29] | [1.3862, 4.6051] | [2, 22] |
| [1.4350, 4.6539] | [8, 26] | [1.6094, 2.9957] | [3, 25] | [0.9808, 4.5643] | [13, 31] |
| [1.7047, 4.4773] | [8, 22] | [1.7917, 3.9889] | [15, 31] | [1.7047, 4.4773] | [0, 30] |
| [0.8472, 5.2417] | [5, 25] | [1.3121, 5.2040] | [7, 19] | [0.7375, 5.5333] | [1, 27] |
| [1.7917, 3.9889] | [11, 19] | [1.6094, 4.8283] | [1, 19] | [2.0794, 3.4657] | [7, 31] |
| [0.9162, 5.0751] | [14, 26] | [1.2527, 4.8362] | [11, 29] | [1.4350, 4.6539] | [9, 29] |
| [0.7731, 4.3567] | [2, 24] | [1.1631, 4.3820] | [2, 18] | [2.0794, 4.2766] | [10, 24] |
| [1.3862, 4.1588] | [12, 30] | [0.7472, 5.1416] | [9, 21] | [0.9162, 4.4998] | [11, 17] |
| [1.2992, 4.8828] | [7, 23] | [1.3862, 4.1588] | [5, 21] | [2.3025, 3.6888] | [1, 19] |
| [0.6286, 4.7874] | [11, 27] | [1.6094, 4.8283] | [8, 26] | [1.2163, 5.3752] | [8, 24] |
| [1.7491, 4.5217] | [3, 25] | [1.5040, 4.2766] | [5, 21] | [0.6359, 5.0304] | [3, 15] |

2. **Model SIM$_2$:** Assume that mid $x_i$'s and spr $x_i$'s are independent random observations from $N(0,\ 0.5)$ and $\chi^2_2$, respectively. Suppose that the independent observed responses $y_{1i}$'s and $y_{2i}$'s come from $b(500, 0.9)$, hence, for all $i = 1,\ \ldots,\ n$, $\eta_i = \log_e(9)$, $o_i = |\log_e(9)|$, and $w_i^M = w_i^S = 45$. Therefore, based on the delta method, mid $\varepsilon \sim N(0, \frac{1}{45})$ and spr $\varepsilon = |TT|$ such that $TT \sim N(0, \frac{1}{45})$, hence, $E\left(\varepsilon^M \mid X^M\right) = 0$ and $E\left(\varepsilon^S \mid X^S\right) = [-\beta_3, \beta_3] = [-0.11893,\ 0.11893]$. For $i = 1,\ \ldots,\ n$, suppose that the model SIM$_2$ is defined as

$$\eta_i\,[1 \pm 0] + o_i\,[0 \pm 1] = \beta_0 \text{ mid } x_i\,[1 \pm 0] + \beta_1 \text{ spr } x_i\,[0 \pm 1] + \beta_2\,[1 \pm 0] + \beta_3\,[0 \pm 1]$$

where $\beta_0 = 5$, $\beta_1 = 1$, $\beta_2 = -3$, and $\beta_3 = 0.11893$. Then suppose that $g_i$ is a random interval determined by using the following model for estimating the parameters of model SIM$_2$:

$$g_i = 5x_i^M - x_i^S - 3\,[1 \pm 0] + e_i = 5x_i^M + x_i^S - 3\,[1 \pm 0] + e_i \qquad i = 1, \ldots,\ n.$$

The model SIGL$_2$ is presented as follow:

$$\mathbf{g}{=}5\ \mathbf{x}^{\mathbf{M}} -\ \mathbf{x}^{\mathbf{S}} - 3\,[1 \pm 0]\ \ \mathbf{1}_{\boldsymbol{n}} + \mathbf{e} = 5\ \mathbf{x}^{\mathbf{M}} +\ \mathbf{x}^{\mathbf{S}} - 3\,[1 \pm 0]\ \ \mathbf{1}_{\boldsymbol{n}} + \mathbf{e} \qquad (4.40)$$

3. **Model SIM$_3$:** Suppose that the independent observed responses $y_{1i}$'s and $y_{2i}$'s come from $Poisson(3)$ and $b(500, .5)$, respectively, so $\eta_i = \log_e(3)$, $o_i = |\log_e(1)|$, $w_i^M = 3$, and $w_i^S = 125$ for all $i = 1, \ldots,\ n$. So, based on the delta method, mid $\varepsilon \sim N(0, \frac{1}{3})$ and spr $\varepsilon = |JJ|$ such that $JJ \sim N(0, \frac{1}{125})$, so $E\left(\varepsilon^M \mid X^M\right) = 0$ and $E\left(\varepsilon^S \mid X^S\right) = [-\beta_3, \beta_3] = [-0.07136,\ 0.07136]$. Meanwhile, suppose that mid $x_i$'s and spr $x_i$'s are independent random observations which come from $Poisson(8)$ and $\chi^2_3$, respectively. For $i = 1, \ldots,\ n$, the model SIM$_3$ is considered as

$$\eta_i\,[1 \pm 0] + o_i\,[0 \pm 1] = \beta_0 \text{ mid } x_i\,[1 \pm 0] + \beta_1 \text{ spr } x_i\,[0 \pm 1] + \beta_2\,[1 \pm 0] + \beta_3\,[0 \pm 1]$$

where $\beta_0 = 1$, $\beta_1 = 4$, $\beta_2 = 0$, and $\beta_3 = 0.07136$. To estimate the parameters of model SIM$_3$, the random interval $g_i$ is defined as

$$g_i = x_i^M - 4x_i^S + e_i = x_i^M + 4x_i^S + e_i \qquad i = 1,\ \ldots,\ n.$$

Hence, the model SIGL$_3$ is written as

$$\mathbf{g}{=}\mathbf{x}^{\mathbf{M}} - 4\ \mathbf{x}^{\mathbf{S}} + \mathbf{e} = \mathbf{x}^{\mathbf{M}} + 4\ \mathbf{x}^{\mathbf{S}} + \mathbf{e} \qquad (4.41)$$

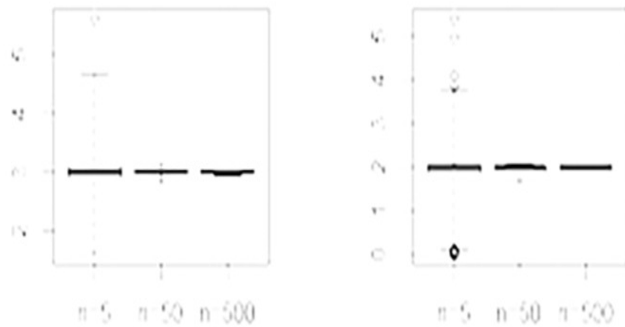

Figure 1: Box plot for $\widehat{\beta}_0$ (left) and $\widehat{\beta}_1$ (right) for model $SIGL_1$ (or model $SIM_1$)

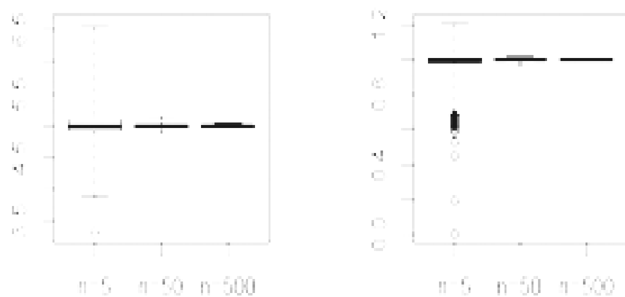

Figure 2: Box plot for $\widehat{\beta}_0$ (left) and $\widehat{\beta}_1$ (right) for model $SIGL_2$ (or model $SIM_2$).

In Table 1, based on the reported experimental results for $10,000$ random samples of different size $n$ from models $\mathrm{SIGL}_1$, $\mathrm{SIGL}_2$, and $\mathrm{SIGL}_3$, the following conclusions can be expressed: First, the asymptotic unbiasedness of the estimators of the parameters of models $\mathrm{SIGL}_1$ (or $\mathrm{SIM}_1$), $\mathrm{SIGL}_2$ (or $\mathrm{SIM}_2$), and $\mathrm{SIGL}_3$ (or $\mathrm{SIM}_3$) are manifest when $n$ increases. Second, as $n$ increases, $\widehat{MSE}\left(\widehat{\beta}_0\right)$, $\widehat{MSE}\left(\widehat{\beta}_1\right)$, $\widehat{MSE}\left(\widehat{\beta}_2\right)$, and $\widehat{MSE}\left(\widehat{\beta}_3\right)$ go to 0. Based on $10,000$ simulated samples of $n$ observations of $(X,\ G)$, the asymptotic unbiasedness and the strong consistency of $\widehat{\beta}_0$ and $\widehat{\beta}_1$ for models $\mathrm{SIGL}_1$, $\mathrm{SIGL}_2$, and $\mathrm{SIGL}_3$ are illustrated as $n$ increases by Figs. 1, 2, and 3, respectively.

Table 1 and Figs. 1, 2, and 3 show the proximity of the estimates to the param-

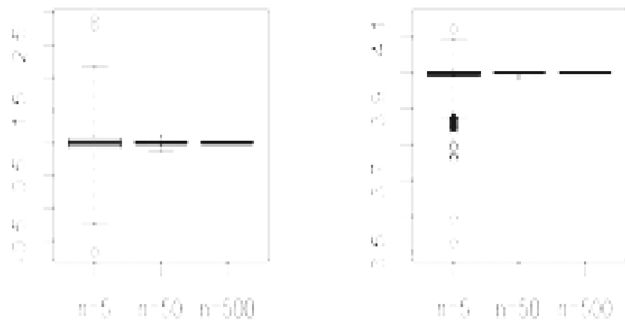

Figure 3: Box plot for $\widehat{\beta}_0$ (left) and $\widehat{\beta}_1$ (right) for model $SIGL_3$ (or model $SIM_3$).

eters of models $\mathrm{SIM}_1$, $\mathrm{SIM}_2$, and $\mathrm{SIM}_3$ as $n$ increases. The empirical performance of the estimation process for models $\mathrm{SIM}_1$, $\mathrm{SIM}_2$, and $\mathrm{SIM}_3$ is illustrated graphically by Fig. 1, Fig. 2, and Fig. 3, respectively. It is also confirmed for all the models in Table 1.

## 4.3 A numerical case study

In this section, an attempt is made to estimate the parameters of a model SIM by analyzing a corresponding model with the structure of model SIGL between $g_i$'s and $x_i$'s in Table 2. We know, the observed responses $y_{1i}$ and $y_{2i}$, $i = 1, \dots,\ 72$, provided in Table 3 came from $Poisson(19)$ and $Poisson(6)$, respectively, hence, $\eta_i = \log_{\mathrm{e}}(19)$ and $o_i = |\log_{\mathrm{e}}(6)|$. In Table 2 or Table 3, mid $x_i$'s and spr $x_i$'s, respectively, came from $Poisson\,(18)$ and $Poisson(8)$. Hence, the aim is to estimate the model

$$\eta_i\,[1 \pm 0] + o_i\,[0 \pm 1] = \beta_0 \text{ mid } x_i\,[1 \pm 0] + \beta_1 \text{ spr } x_i\,[0 \pm 1] + \beta_2\,[1 \pm 0] + \beta_3\,[0 \pm 1]\,,$$

$i = 1,\ \dots,\ 72$, when $\beta_0$, $\beta_1$, $\beta_2$, and $\beta_3$ are unknown values.

In Table 2, $g_i = [\ \log_{\mathrm{e}}(y_{1i}) - |\log_{\mathrm{e}}(y_{2i})|\,,\ \ \log_{\mathrm{e}}(y_{1i}) + |\log_{\mathrm{e}}(y_{2i})|\ ]$ is defined as an initial estimate of $\eta_i\,[1 \pm 0] + o_i\,[0 \pm 1] = [\eta_i - o_i, \eta_i + o_i]$, $i = 1,\ \dots,\ 72$. Based on the data set provided in Table 2, using the proposed estimation process in Section 3, will obtain a suitable estimation of the model

$$\mathbf{g} {=} \beta_0\ \mathbf{x^M} {+} \beta_1\ \mathbf{x^S} {+} \beta_2\,[1 \pm 0]\ \ \mathbf{1}_{\boldsymbol{n}} {+} \mathbf{e}$$

(or the above model SIM). Hence, a good estimation of the model is provided based on the set $(\widehat{\beta}_0,\ \widehat{\beta}_1) \in U = \mathbb{R} \times [0, \hat{u}_0]$. Hence, the estimates $\widehat{\beta}_0 = 0.1605577$,

Table 3: Observed data set for model building.

| $y_2$ | $y_1$ | $x$ | $y_2$ | $y_1$ | $x$ | $y_2$ | $y_1$ | $x$ |
|---|---|---|---|---|---|---|---|---|
| 10 | 26 | [15, 35] | 2 | 14 | [10, 20] | 4 | 28 | [2, 26] |
| 3 | 18 | [12, 30] | 8 | 16 | [6, 30] | 5 | 19 | [6, 24] |
| 4 | 21 | [6, 22] | 5 | 23 | [7, 27] | 11 | 15 | [13, 33] |
| 6 | 26 | [8, 32] | 9 | 14 | [11, 19] | 8 | 22 | [17, 25] |
| 6 | 22 | [12, 22] | 6 | 14 | [9, 23] | 9 | 23 | [6, 26] |
| 2 | 13 | [6, 26] | 5 | 23 | [5, 29] | 8 | 18 | [15, 31] |
| 7 | 18 | [17, 29] | 5 | 25 | [7, 27] | 2 | 14 | [9, 29] |
| 7 | 8 | [16, 32] | 7 | 22 | [6, 18] | 5 | 21 | [10, 22] |
| 9 | 19 | [3, 21] | 4 | 13 | [3, 23] | 6 | 17 | [8, 26] |
| 5 | 12 | [3, 21] | 11 | 25 | [9, 25] | 2 | 12 | [15, 31] |
| 6 | 23 | [12, 28] | 6 | 18 | [5, 11] | 4 | 13 | [15, 35] |
| 6 | 16 | [13, 39] | 4 | 21 | [19, 29] | 4 | 13 | [12, 24] |
| 6 | 24 | [10, 22] | 5 | 28 | [18, 26] | 9 | 13 | [22, 32] |
| 4 | 19 | [13, 19] | 5 | 24 | [11, 29] | 5 | 20 | [2, 22] |
| 5 | 21 | [8, 26] | 2 | 10 | [3, 25] | 6 | 16 | [13, 31] |
| 4 | 22 | [8, 22] | 3 | 18 | [15, 31] | 4 | 22 | [0, 30] |
| 9 | 21 | [5, 25] | 7 | 26 | [7, 19] | 11 | 23 | [1, 27] |
| 3 | 18 | [11, 19] | 5 | 25 | [1, 19] | 2 | 16 | [7, 31] |
| 8 | 20 | [14, 26] | 6 | 21 | [11, 29] | 5 | 21 | [9, 29] |
| 6 | 13 | [2, 24] | 5 | 16 | [2, 18] | 3 | 24 | [10, 24] |
| 4 | 16 | [12, 30] | 9 | 19 | [9, 21] | 6 | 15 | [11, 17] |
| 6 | 22 | [7, 23] | 4 | 16 | [5, 21] | 2 | 20 | [1, 19] |
| 8 | 15 | [11, 27] | 5 | 25 | [8, 26] | 8 | 27 | [8, 24] |
| 4 | 23 | [3, 25] | 4 | 18 | [5, 21] | 9 | 17 | [3, 15] |

$\widehat{\beta}_1 = 0.05776227$, $\widehat{\beta}_2 = 0.1803286$, and $\widehat{\beta}_3 = 1.164749$ are obtained. The final estimate of $\eta_i\,[1 \pm 0] + o_i\,[0 \pm 1]$, $i = 1, \ldots,\ 72$, is a random interval determined by using the following model:

$$[\widehat{\eta}_i - \hat{o}_i, \widehat{\eta}_i + \hat{o}_i] = 0.1605577\, x_i^M + 0.05776227\, x_i^S + 0.1803286\,[1 \pm 0] + 1.164749\,[0 \pm 1]\,.$$

Also, the estimated interval model between $\mathbf{g}$ and $\mathbf{x}$ is as follows:

$$\widehat{\mathbf{g}} = 0.1605577\ \mathbf{x^M} + 0.05776227\ \mathbf{x^S} + [-0.9844204,\ 1.345078]\ \mathbf{1}_{\boldsymbol{n}}. \qquad (4.42)$$

For instance, the predicted intervals

$\hat{g}_9 = [1.225,\ 4.594]$, $\hat{g}_{52} = [1.534,\ 5.249]$, and $\hat{g}_{63} = [1.225,\ 4.594]$,

respectively, are obtained for the units $(x_9, g_9)$, $(x_{52}, g_{52})$, and $(x_{63}, g_{63})$ given in Table 2 based on (37).

## 5. Conclusion

Based on the interval arithmetic, for the first time, an interval generalized linear model was introduced for interval-valued data in this paper. This model was called model SIM. Then a model was proposed to estimate the parameters of model SIM based on the interval arithmetic. The model was called model SIGL. The estimation procedure of the parameters of model SIGL was applied as the estimation procedure of the parameters of model SIM. In model SIGL, observations of the interval-valued variables were defined as intervals. The estimators of the parameters of model SIGL (or model SIM) were obtained with respect to some good properties of the space of intervals, which are provided by the arithmetic and the metric described in this research. Due to the semi-linearity of the interval space, the estimates of the parameters of model SIGL (or model SIM) were considered in a set of possible values associated with the interval nature of the variables. The theoretical adequacy of the estimators of model SIGL (or model SIM) was studied and demonstrated. The empirical validation of the estimation procedure for model SIM was investigated by studying the proximity of the estimates to the parameters of model SIM (or model SIGL) using a Monte Carlo simulation.

In this paper, the distributions of the independent random response variables were members of the exponential family of distributions; for instance, the normal, binomial, Poisson, geometric, negative binomial, exponential, gamma, and inverse normal.

Model SIM can be used when the research is associated with different facts (grouping, censoring, uncertainty in the measure, and so on), or when the study is focused just on interval-valued characteristics (fluctuations, ranges of variation, etc.).

## Acknowledgements

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

# Appendix

**Proofs of the LS estimators given in (23) and (24).** Using the first matrix form in (22), we can provide

$$\left(\begin{pmatrix} 1 & x_1^M \\ \vdots & \vdots \\ 1 & x_n^M \end{pmatrix}' \begin{pmatrix} 1 & x_1^M \\ \vdots & \vdots \\ 1 & x_n^M \end{pmatrix}\right)^{-1} \begin{pmatrix} 1 & x_1^M \\ \vdots & \vdots \\ 1 & x_n^M \end{pmatrix}' \begin{pmatrix} g_1^M \\ \vdots \\ g_n^M \end{pmatrix} = \begin{pmatrix} \widehat{\beta}_{2,\ LS} \\ \widehat{\beta}_{0,\ LS} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\left(\sum_{i=1}^n (\text{mid } x_i)^2\right)\left(\sum_{i=1}^n (\text{mid } g_i)\right) - \left(\sum_{i=1}^n (\text{mid } x_i)\right)\left(\sum_{i=1}^n (\text{mid } x_i)(\text{mid } g_i)\right)}{n\left(\sum_{i=1}^n (\text{mid } x_i)^2\right) - \left(\sum_{i=1}^n (\text{mid } x_i)\right)^2} \\ \frac{n\left(\sum_{i=1}^n (\text{mid } x_i)(\text{mid } g_i)\right) - \left(\sum_{i=1}^n (\text{mid } x_i)\right)\left(\sum_{i=1}^n (\text{mid } g_i)\right)}{n\left(\sum_{i=1}^n (\text{mid } x_i)^2\right) - \left(\sum_{i=1}^n (\text{mid } x_i)\right)^2} \end{pmatrix}.$$

Therefore, $\widehat{\beta}_{2,\ LS}$ and $\widehat{\beta}_{0,\ LS}$ in (23) are obtained. Using the second matrix form in (22), $\widehat{\beta}_{3,\ LS}$ and $\widehat{\beta}_{1,\ LS}$ given in (23) are concluded analogously. Based on $\widehat{\beta}_{0,\ LS}$ given in (23), $\widehat{\beta}_{0,\ LS}$ given in (24) is obtained as follows:

$$\begin{aligned}
\widehat{\beta}_{0,\ LS} &= \frac{\left(n\left(\sum_{i=1}^n (\text{mid } x_i)(\text{mid } g_i)\right) - \left(\sum_{i=1}^n (\text{mid } x_i)\right)\left(\sum_{i=1}^n (\text{mid } g_i)\right)\right)/n}{\left(n\left(\sum_{i=1}^n (\text{mid } x_i)^2\right) - \left(\sum_{i=1}^n (\text{mid } x_i)\right)^2\right)/n} \\
&= \frac{\left(\sum_{i=1}^n (\text{mid } x_i)(\text{mid } g_i)\right) - 2\frac{\left(\sum_{i=1}^n (\text{mid } x_i)\right)\left(\sum_{i=1}^n (\text{mid } g_i)\right)}{n} + \frac{\left(\sum_{i=1}^n (\text{mid } x_i)\right)\left(\sum_{i=1}^n (\text{mid } g_i)\right)}{n}}{\left(\sum_{i=1}^n (\text{mid } x_i)^2\right) - 2\frac{\left(\sum_{i=1}^n (\text{mid } x_i)\right)^2}{n} + \frac{\left(\sum_{i=1}^n (\text{mid } x_i)\right)^2}{n}} \\
&= \frac{\sum_{i=1}^n \left((\text{mid } x_i)(\text{mid } g_i) - (\text{mid } x_i)\left(\overline{\text{mid } G}\right) - (\text{mid } g_i)\left(\overline{\text{mid } X}\right) + \left(\overline{\text{mid } G}\right)\left(\overline{\text{mid } X}\right)\right)}{\sum_{i=1}^n \left((\text{mid } x_i)^2 - 2(\text{mid } x_i)\left(\overline{\text{mid } X}_{WM}\right) + \left(\overline{\text{mid } X}_{WM}\right)^2\right)} \\
&= \frac{\left(\sum_{i=1}^n \left(\text{mid } x_i - \overline{\text{mid } X}\right)\left(\text{mid } g_i - \overline{\text{mid } G}\right)\right)/n}{\left(\sum_{i=1}^n \left(\text{mid } x_i - \overline{\text{mid } X}\right)^2\right)/n} = \frac{\widehat{\sigma}(\text{mid } X,\ \ \text{mid } G)}{\widehat{\sigma}^2(\text{mid } X)} \\
&= \frac{\widehat{\sigma}\left(X^M,\ \ G^M\right)}{\widehat{\sigma}^2\left(X^M\right)}.
\end{aligned}$$

Based on $\widehat{\beta}_{1,\ LS}$, $\widehat{\beta}_{2,\ LS}$, and $\widehat{\beta}_{3,\ LS}$ provided in (23), respectively, the estimators $\widehat{\beta}_{1,\ LS}$, $\widehat{\beta}_{2,\ LS}$, and $\widehat{\beta}_{3,\ LS}$ provided in (24) are obtained analogously.

**Proof of Theorem 4.1.** The strong consistency of $\widehat{\beta}_{0,\ LS}$ given in (24) for the parameter $\beta_0 = \frac{\sigma(\text{mid } X,\ \text{mid } G)}{\sigma^2(\text{mid } X)}$ of the first model given in (15) (or the model $\eta_i = \beta_0$ mid $x_i + \beta_2$, $i = 1, \ldots, n$) will result the strong consistency of $\widehat{\beta}_0$ given in (29) with respect to the parameter $\beta_0$.

We are going to prove the strong consistency of $\widehat{\beta}_1$ given in (30) with respect to $\beta_1$, when the sample size $n$ tends to infinity. First, we will demonstrate that

$$T_n = \max\left\{0, \frac{\widehat{\sigma}\,(\text{spr } X,\ \text{spr } G)}{\widehat{\sigma}^2\,(\text{spr } X)}\right\} = \max\left\{0, \frac{\widehat{\sigma}\left(X^S,\ G^S\right)}{\widehat{\sigma}^2\left(X^S\right)}\right\} \overset{n\to\infty}{\longrightarrow} \beta_1 \quad \text{a.s.} - \ [\text{P}].$$

When $n$ tends to infinity, based on the strong consistency of $\widehat{\sigma}\,(\text{spr } X,\ \text{spr } G)$, $\widehat{\sigma}\left(X^S,\ G^S\right)$, $\widehat{\sigma}^2\,(\text{spr } X)$, and $\widehat{\sigma}^2\left(X^S\right)$ with respect to $\sigma\,(\text{spr } X,\ \text{spr } G)$, $\sigma\left(X^S, G^S\right)$, $\sigma^2\,(\text{spr } X)$, and $\sigma^2\left(X^S\right)$, respectively, the following expression is provided:

$$\widehat{\beta}_{1,\ LS} = \frac{\widehat{\sigma}\,(\text{spr } X,\ \text{spr } G)}{\widehat{\sigma}^2\,(\text{spr } X)} = \frac{\widehat{\sigma}\left(X^S,\ G^S\right)}{\widehat{\sigma}^2\left(X^S\right)} \overset{n\to\infty}{\longrightarrow} \beta_1 = \frac{\sigma\,(\text{spr } X,\ \text{spr } G)}{\sigma^2\,(\text{spr } X)} = \frac{\sigma\left(X^S,\ G^S\right)}{\sigma^2\left(X^S\right)} \quad \text{a.s.} - \ [\text{P}].$$

By considering the real continuous mapping max $\{0, \bullet\}$, the following expression can be obtained:

$$T_n = \max\left\{0,\ \widehat{\beta}_{1,LS}\right\} \overset{n\to\infty}{\longrightarrow} \max\ \{0,\ \beta_1\} \quad \text{a.s.} \tag{5.43}$$

By considering the values of $\beta_1$, the expression (A.1) can also be checked as follows:

The expression $\max\{0, \beta_1\} = \beta_1$ can be concluded when $\beta_1 > 0$ or equivalently $\sigma\,(\text{spr } X, \text{spr } G) > 0$.

The conclusion $\max\{0, \beta_1\} = 0$ can be obtained when $\beta_1 = 0$ or equivalently $\sigma\,(\text{spr } X, \text{spr } G) = 0$.

The convergence in (38) is concluded based on the two cases as follows:

$$T_n \overset{n\to\infty}{\longrightarrow} \beta_1 \qquad \text{a.s. [P]}. \tag{5.44}$$

We prove $\widehat{\beta}_1 = \min\{\hat{u}_0,\ T_n\} \overset{n\to\infty}{\longrightarrow} \beta_1 = \min\{u_0, \beta_1\}$ (see below).

The result $\min\{u_0,\ T_n\} \leq \min\{\hat{u}_0,\ T_n\} = \widehat{\beta}_1$ can be obtained based on $\widehat{\beta}_1 \leq T_n$, and also $u_0 \leq \hat{u}_0$, where $\hat{u}_0$ is as the sample of $u_0$. Thus,

$$\min\{u_0, T_n\} \leq \widehat{\beta}_1 \leq T_n. \tag{5.45}$$

By considering spr $x_i$, spr $e_i \geq 0$ for $i = 1, \ldots, n$, and $\beta_1 \geq 0$ in the second model given in (15), the conclusion $\min\{u_0, \beta_1\} = \beta_1$ can be expressed.

The convergence $\min\ \{u_0, T_n\} \overset{n\to\infty}{\longrightarrow} \min\ \{u_0, \beta_1\}\ = \beta_1$ a.s.- [P] can be concluded using the real continuous mapping $\min\{u_0,\ \bullet\}$ . So the convergence $\widehat{\beta}_1 \overset{n\to\infty}{\longrightarrow} \beta_1$ a.s.- [P] is proven with respect to the convergence $\min\ \{u_0, T_n\} \overset{n\to\infty}{\longrightarrow} \min\ \{u_0, \beta_1\}\ = \beta_1$ a.s.- [P], the conclusion given in (39), and using the sandwich rule in (40).

Finally, the strong consistency of the estimator

$$\widehat{\beta}_4 = \left(\overline{G^M} + \overline{G^S}\right) - \left(\widehat{\beta}_0\ \overline{X^M} + \widehat{\beta}_1 \overline{X^S}\right)$$

with respect to the parameter $\beta_4 = \left(E\left(G^M\right) + E\left(G^S\right)\right) - \left(\beta_0\ E\left(X^M\right) + \beta_1\ E(X^S)\right)$ follows from the corresponding convergences of $\widehat{\beta}_0$ and $\widehat{\beta}_1$, together with the *strong law of large numbers* for the sample mean of an interval random set (see Artstein and Vitale, 1975).

# References

Artstein, Z.,Vitale, R.A., (1975). A strong law of large numbers for random compact sets. *The Annals of Probability*, **3(5)**, 879-882.

Bertrand, P., Goupil, F., (2000). *Descriptive statistics for symbolic data Analysis of symbolic data*, 106-124, Springer.

Billard, L., and Diday, E., (2000).*Regression analysis for interval-valued data Data Analysis, Classification, and Related Methods.* (pp. 369-374): Springer.

Billard, L., (2006). Symbolic data analysis: what is it? *Compstat 2006-Proceedings in Computational Statistics* (pp. 261-269): Springer.

Billard, L., (2008). Sample covariance functions for complex quantitative data. *Paper presented at the Proceedings World Congress International Association Statistical Computing.*

Billard, L., (2011). Brief overview of symbolic data and analytic issues. *Statistical Analysis and Data Mining: The ASA Data Science Journal*, **4(2)**, 149-156.

Calle, M.L., and Gomez, G., (2001). Nonparametric Bayesian estimation from interval-censored data using Monte Carlo methods. *Journal of Statistical Planning and Inference*, **98**, 73–87.

Chesher, A., and Irish, M., (1987). Residual analysis in the grouped and censored normal linear model. *Journal of Econometrics*, **34**, 33–61.

Diamond, P., (1990). Least squares fitting of compact set-valued data. *Journal of Mathematical Analysis and Applications*, **147(2)**, 351-362.

de Carvalho, F.A.T., Lima Neto, E.A., and Tenorio, C.P., (2004). A new method to fit a linear regression model for interval-valued data. *KI 2004: Advances in Artificial Intelligence*, 295-306.

de A. Lima Neto E., de Carvalho F.A.T., and Tenorio C.P., (2004). Univariate and multivariate linear regression methods to predict interval-valued features.*Paper presented at the Australasian Joint Conference on Artificial Intelligence.*

de A. Lima Neto E., de A.T. de Carvalho F., and Freire E.S., (2005). Applying constrained linear regression models to predict interval-valued data. *Paper presented at the Annual Conference on Artificial Intelligence.*

de A. Lima Neto E., de A.T. de Carvalho F., (2008). Centre and range method for fitting a linear regression model to symbolic interval data. *Computational Statistics & Data Analysis*, **52(3)**, 1500-1515.

de A. Lima Neto E., Cordeiro, G.M., de A.T. de Carvalho F., dos Anjos, U.U.,and da Costa, A.G., (2009). Bivariate generalized linear model for interval-valued variables. *Paper presented at the 2009 International Joint Conference on Neural Networks.*

de A. Lima Neto E.,and de A.T. de Carvalho F., (2010). Constrained linear regression models for symbolic interval-valued variables. *Computational Statistics & Data Analysis*, **54(2)**, 333-347.

Gil, M.Á., López-García, M.T., Lubiano, M.A., and Montenegro, M., (2001). Regression and correlation analyses of a linear relation between random intervals.*test*, **10(1)**, 183-201.

M.Á., Lubiano, M.A., Montenegro, M., and López, M.T., (2002). Least squares fitting of an affine function and strength of association for interval-valued data. *Metrika*, **56(2)**, 97-111.

Gil, M.Á., González-Rodríguez, G.,Colubi, A., and Montenegro, M., (2007). Testing linear independence in linear models with interval-valued data. *Computational Statistics & Data Analysis*, **51(6)**, 3002-3015.

Gonalez-Rodriguez, G., Blanco, A., Corral, N., and Colubi, A., (2007). Least squares estimation of linear regression models for convex compact random sets. *Advances in Data Analysis and Classification* , **1**, 67–81.

Hong, H., and Tamer, E., (2003). Inference in censored models with endogenous regressors.*Econometrica*, **71(3)**, 905–932.

Huber, C., Solev, V., and Vonta, F., (2009). Interval censored and truncated data: rate of convergence of NPMLE of the density. *Journal of Statistical Planning and Inference* , **139**, 1734–1749.

Jahanshahloo, G.R., Hosseinzadeh Lotfi, F., Rostamy Malkhalifeh, M., and Ahadzadeh Namin, M., (2008). A generalized model for data envelopment analysis with interval data. *Applied Mathematical Modelling*, **33**, 3237–3244.

Manski, C.F., and Tamer, E., (2002). Inference on regressions with interval data on a regressor or outcome. *Econometrica*, **70(2)**, 519–546.

Nelder, J.A., Wedderburn, R.W.M., (1972). Generalized linear models. *Journal of the Royal Statistical Society, Series A*, **135**, 370-384.

Rivero, C., and Valdes, T., (2008). An algorithm for robust linear estimation with grouped data. *Computational Statistics & Data Analysis*, **53**, 255–271.

Wang, H., Guan, R., and Wu, J., (2012). Linear regression of interval-valued data based on complete information in hypercubes. *Journal of Systems Science and Systems Engineering*, **21(4)**, 422-442.

Zhao, X., Lim, H.J., and Sun, J., (2005). Estimating equation approach for regression analysis of failure time data in the presence of interval-censoring. *Journal of Statistical Planning and Inference*, **129**, 145–157.