Journal of Mathematics and Modeling in Finance (JMMF)
Vol. 2, No. 1, Winter & Spring 2022
Research paper

A Numerical solution for the new model of time-fractional bond pricing: Using a multiquadric approximation method

Sedighe Sharifian[1], Ali R. Soheili[2], Abdolsadeh Neisy[3]

[1] Department of Applied Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran
sedighe.sharifian1985@gmail.com

[2] Department of Applied Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran
soheili@um.ac.ir

[3] Department of Mathematics, Faculty of Statistics, Mathematics & Computer, Allameh Tabataba'i University, Tehran Iran

a_neisy@atu.ac.ir

Abstract:
The bond market is an important part of the financial markets. The coupon bonds are issued by companies or banks for increasing capital, and the interest is paid by banks or companies, periodically. In terms of maturities, bonds are divided into three categories as follows: short term, medium term, and long term In this paper, we model the fractional bond pricing under fractional stochastic diferential equation. We implement the multiquadric approximation for solving the fractional bond pricing equation. The equation is discretized in the time direction base on modified Riemann-Liouville derivative and finite diference methods and is approximated by using the multiquadric approximation method in the space direction which achives the semi-discrete solution. We investigate the unconditional stability and convergence of the proposed method. The method presented in the article has been implemented on two examples with different values, which confirm the results of the effectiveness of the method and show that appropriate results can be obtained with the MQ method. it should be noted that all calculations were done with the help of matlab software. Numerical results demonstrate the efficiency and ability of the presented method.

Keywords: Fractional derivative, Fractional interest rate, Time-fractional bond pricing, Multiquadric approximation method
Mathematics Subject Classification: 65M70, 91G80

1 Introduction

Fractional Differential Calculus (FDC) was born in the 17th century and its initial discussions were related to the works of Leibniz, Euler, Lagrange, Laplace, Abel, Liouville, Riemann, and others. In recent decades, the fractional differential equations have been considered in different fields such as fluid flow, engineering, electromagnetics, economics, and finance [1]. They have been also used in many different issues, such as telegraph, diffusion, and the diffusion–wave fractional equa-

[1]Corresponding author
Received: 06/06/2022 Accepted: 08/09/2022
http://dx.doi.org/10.22054/jmmf.2022.68274.1056

tion [2–4, 4, 5]. In finance, Wyss (2000) provided the time–fractional Black–Scholes equation for pricing European call option, and Jumarie (2010) presented time–fractional and space–fractional Black–Scholes equations [6, 7]. Many researchers have solved the fractional Black–Scholes equation and examined its stability and convergence [8–10, 12, 13]. In this paper, we intend to present a model for the fractional bond pricing.

The bond market is an important part of the financial markets. The coupon bonds are issued by companies or banks for increasing capital, and the interest is paid by banks or companies, periodically. When, the bond has no interest, it will be a zero coupon bond. In terms of maturities, bonds are divided into three categories as follows: short term, medium term, and long term [24]. In the classical model of bond pricing, the short interest rate is a function of standard Brownian motion. This motion lacks long–range dependence and loses its previous condition. The existence of a long–range dependence in asset returns has important applications in examining the market, bond pricing, and selecting asset portfolio [14]. Mandelbrot was the first one that to offers the idea of the long–range dependence in asset returns [15].

The fractional Brownian motion has stationary increments and long–range dependence which is compatible with market reality. Existence of this process in the fractional pricing model will make the price more realistic [12]. According to the mentioned contents in this work, we use the fractional interest rate instead of the standard interest rate. Therefore, the fractional bond pricing equation becomes a generalized classical bond pricing equation by replacing the fractional interest rate in the standard interest rate.

The fractional bond pricing model can be solved by different methods. In this paper, we use the multi–quadric (MQ) approximation method because this method of solution works well, in particular when the data points are scattered. This method was introduced by Hardy (1971). He did not relate the base function to the coordinates (space) of the points and set it as a function of distance (radial) by using the concept of norm. The proposed function was $\varphi(p) = \sqrt{(p^2 + C^2)}$. It is the best method for the numerical solution of ordinary differential equations and partial differential equations [21, 23]. We will describe some advantages of this method in section 3.1.

The paper is organized as follows: In section 2, we model the fractional bond pricing. In section 3, first, the multiquadric approximation method is introduced. Then, we describe the time discretization process and a boundary value problem (BVP) is obtained which is solved by the multiquadric approximation method. In section 4, we investigate the unconditional stability and convergence. In section 5, the numerical result is described. The last section includes conclusion and recommendation for future study.

2 The model

The classical short interest rate model described by the following stochastic differential equation [13, 24]:

$$dr = \mu(r,t)dt + \rho(r,t)dW_t,$$

where dW_t is the standard Brownian motion and $\mu(r,t)$ and $\rho(r,t)$ are drift and variance parameters, respectively. In the fractional condition, such as the fractional underlying asset in [7, 25], suppose that $r = r_t = r(t)$ follows the fractional stochastic differential equation:

$$dr = \mu(r,t)dt + \rho(r,t)w(t).(dt)^{\frac{\alpha}{2}}, \tag{1}$$

where $w(t)$ is a normalized Gaussian white noise with unit variance and zero mean. Unfortunately, the exponents 1 and $\frac{\alpha}{2}$ of dt are not consistent with fractional Taylor's series. So, we use two formulas to solve this problem as follows:

$$d^{\alpha}r = \Gamma(1+\alpha)dr, \qquad 0 < \alpha \leqslant 1, \tag{2}$$

$$\frac{d^{\alpha}r}{dr^{\alpha}} = \frac{1}{\Gamma(2-\alpha)}r^{(1-\alpha)}, \qquad 0 < \alpha \leqslant 1. \tag{3}$$

For the proof of (1) and (2), see [25]. By substituting (2) in (1), we have

$$dr = \mu(r,t)(\Gamma(1+\alpha))^{(-1)}d^{\alpha}t + \rho(r,t)\omega(t)(dt)^{\frac{\alpha}{2}}; \tag{4}$$

now, by substituting (3) in (4), we obtain

$$dr = \frac{1}{\Gamma(1+\alpha)\Gamma(2-\alpha)}\mu(r,t)t^{(1-\alpha)}(dt)^{\alpha} + \rho(r,t)\omega(t)(dt)^{\frac{\alpha}{2}}, \qquad 0 < \alpha \leqslant 1. \tag{5}$$

According to Lemma 3.2 in [7], equation (5) is consistent with fractional Taylor's series. By combining (2) and (3), we obtain

$$dr = \frac{r^{(1-\alpha)}}{\Gamma(1+\alpha)\Gamma(2-\alpha)}(dr)^{\alpha}, \qquad 0 < \alpha \leqslant 1. \tag{6}$$

Now, we assume that the bond pricing $V(r,t)$ is twice differentiable with respect to r and has a fractional derivative of order α with respect to t.

When the interest rate follows equation (1), the bond has the price of $V(r,t)$. According to the assumptions, fractional Taylor's formula, and Itô's lemma, it can be written as follows [7]:

$$dV = \frac{1}{\Gamma(1+\alpha)}V_t^{(\alpha)}(dt)^{\alpha} + V_r dr + \frac{1}{2}V_{rr}(dr)^2; \tag{7}$$

by replacing the equation (7) by (1), we have

$$dV = \frac{1}{\Gamma(1+\alpha)}V_t^{(\alpha)}(dt)^{\alpha} + V_r\mu dt + V_r\rho w(t).(dt)^{(\frac{\alpha}{2})} + \frac{1}{2}V_{rr}\rho^2(dt)^{\alpha}. \tag{8}$$

It cannot be used to hedge with the bond because the interest rate is not a traded security. So, we use different maturities (T_1 and T_2) for hedging bonds. We buy a bond of value V_1 with a maturity T_1 and sell the another bond of value V_2 with a maturity T_2, where $V_1 = V(r,t;\Pi_1)$ and $V_2 = V(r,t;\Pi_2)$ [24].

Therefore, the portfolio (Π) value is calculated as follows [13]:

$$\Pi = V_1 - \Delta V_2, \tag{9}$$

where Δ is a covering risk.

The change of portfolio value in time dt is as follows:

$$d\Pi = dV_1 - \Delta dV_2. \tag{10}$$

Then we substitute (8) into (10),

$$\begin{aligned} d\Pi &= \left(\tfrac{1}{\Gamma(1+\alpha)} V_{1t}^{(\alpha)}(dt)^\alpha + V_{1r}\mu dt + V_{1r}\rho w(t)(dt)^{(\frac{\alpha}{2})} + \tfrac{1}{2}V_{1rr}\rho^2(dt)^\alpha\right) \\ &\quad -\Delta\left(\tfrac{1}{\Gamma(1+\alpha)} V_{2t}^{(\alpha)}(dt)^\alpha + V_{2r}\mu dt + V_{2r}\rho w(t)(dt)^{(\frac{\alpha}{2})} + \tfrac{1}{2}V_{2rr}\rho^2(dt)^\alpha\right). \end{aligned} \tag{11}$$

By selecting $\Delta = \frac{V_{1r}}{V_{2r}}$ in (11), the stochastic component is eliminated:

$$\begin{aligned} d\Pi &= \tfrac{1}{\Gamma(1+\alpha)} V_{1t}^{(\alpha)}(dt)^\alpha + \tfrac{1}{2}V_{1rr}\rho^2(dt)^\alpha \\ &\quad -\tfrac{V_{1r}}{V_{2r}}\tfrac{1}{\Gamma(1+\alpha)} V_{2t}^{(\alpha)}(dt)^\alpha - \tfrac{1}{2}\tfrac{V_{1r}}{V_{2r}}V_{2rr}\rho^2(dt)^\alpha. \end{aligned} \tag{12}$$

Since the portfolio is instantaneously riskless, we should obtain the riskless short interest rate, which is as follows:

$$d\Pi = r\Pi dt = r(V_1 - \frac{V_{1r}}{V_{2r}}V_2)dt. \tag{13}$$

Replacing (6) into (41), implies

$$d\Pi = r\left(V_1 - \frac{V_{1r}}{V_{2r}}V_2\right)\frac{t^{1-\alpha}}{\Gamma(1+\alpha)\Gamma(2-\alpha)}(dt)^\alpha, \tag{14}$$

now, by replacing (12) by (14), we obtain

$$\begin{aligned} &\left(\tfrac{1}{\Gamma(1+\alpha)} V_{1t}^{(\alpha)}(dt)^\alpha + \tfrac{1}{2}V_{1rr}\rho^2(dt)^\alpha - r\tfrac{t^{(1-\alpha)}}{\Gamma(1+\alpha)\Gamma(2-\alpha)}V_1(dt)^\alpha\right) \\ &+\left(-\tfrac{V_{1r}}{V_{2r}}\tfrac{1}{\Gamma(1+\alpha)} V_{2t}^{(\alpha)}(dt)^\alpha - \tfrac{1}{2}\tfrac{V_{1r}}{V_{2r}}V_{2rr}\rho^2(dt)^\alpha + r\tfrac{t^{1-\alpha}}{\Gamma(1+\alpha)\Gamma(2-\alpha)}\tfrac{V_{1r}}{V_{2r}}V_2(dt)^\alpha\right) = 0. \end{aligned} \tag{15}$$

Multiplying (15) by $\frac{\Gamma(1+\alpha)}{(dt)^\alpha}$ yields

$$\begin{aligned} &\left(V_{1t}^{(\alpha)} + \tfrac{\Gamma(1+\alpha)}{2}V_{1rr}\rho^2 - r\tfrac{t^{(1-\alpha)}}{\Gamma(2-\alpha)}V_1\right) \\ &= \tfrac{V_{1r}}{V_{2r}}\left(V_{2t}^{(\alpha)} + \tfrac{\Gamma(1+\alpha)}{2}V_{2rr}\rho^2 - r\tfrac{t^{(1-\alpha)}}{\Gamma(2-\alpha)}V_2\right), \end{aligned} \tag{16}$$

with gathering together all V_1 terms on the left-hand side and all V_2 terms on the right-hand side, we find

$$\frac{\left(V_{1t}^{(\alpha)} + \frac{\Gamma(1+\alpha)}{2} V_{1rr}\rho^2 - r\frac{t^{(1-\alpha)}}{\Gamma(2-\alpha)} V_1\right)}{V_{1r}} = \frac{\left(V_{2t}^{(\alpha)} + \frac{\Gamma(1+\alpha)}{2} V_{2rr}\rho^2 - r\frac{t^{(1-\alpha)}}{\Gamma(2-\alpha)} V_2\right)}{V_{2r}}. \tag{17}$$

Now both sides of (17) are independent of the maturity time, by assuming $a(r,t)$ as follows:

$$\frac{\left(V_{t}^{(\alpha)} + \frac{\Gamma(1+\alpha)}{2} V_{rr}\rho^2 - r\frac{t^{(1-\alpha)}}{\Gamma(2-\alpha)} V\right)}{\left(\frac{\partial V}{\partial r}\right)} = a(r,t), \tag{18}$$

according to [24]

$$a(r,t) = \rho\lambda - \mu, \tag{19}$$

where λ is the market price of risk. So, we have the fractional zero-coupon bond pricing equation

$$V_t^{(\alpha)} + \frac{\Gamma(1+\alpha)}{2}\rho^2 V_{rr} - r\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} V + (\mu - \lambda\rho)V_r = 0, \tag{20}$$

where $0 < t < T$.

In equation (20), we can use different interest rate models for μ and ρ, which lead to different dynamics of instant interest rate. In this article we use the Vasicek mean-reversion interest rate which is corresponding to $\mu(r,t) = \nu(\gamma - r)$ and $\rho(r,t) = \rho$ in (1) [13]. Thus the fractional bond pricing model can be obtained as follows:

$$V_t^{(\alpha)} + \frac{\Gamma(1+\alpha)}{2}\rho^2 V_{rr} - r\frac{t^{(1-\alpha)}}{\Gamma(2-\alpha)} V + (\nu(\gamma - r) - \lambda\rho)V_r = 0, \tag{21}$$

where $0 < t < T$. We need initial and boundary conditions to solve (21). According to [26], we have

$$\begin{cases} V(r,T) = 1, & 0 < r < R, \quad t = T, \\ \frac{\partial V(0,t)}{\partial t} + \mu(0)\frac{\partial V(0,t)}{\partial r} = 0, & r = 0, \quad 0 < t < T, \\ \frac{\partial V(R,t)}{\partial t} + \mu(R)\frac{\partial V(R,t)}{\partial r} = RV, & r = R, \quad 0 < t < T. \end{cases} \tag{22}$$

Equation (21) with conditions of (22) can be solved by different methods. In the next section, we solve this equation by the multiquadric-RBF approximation method.

3 Numerical method

In this section, we first define the multi-quadric approximation method. Then, equation (21) with conditions of (22) is solved.

3.1 Multi-quadric approximation method

The MQ approximation method is a suitable method for numerical solution of the initial and boundary value problem. Some advantages of this method are as follows:

• This method is very flexible because it does not depend on the positions of the points and it can be used as an interpolation method for scattered points.

• It depends on the shape parameter, and we can obtain a convergence of high accuracy for the pricing problem by setting this parameter and obtaining its different values.

We assume that each function can be expanded as a finite series of upper hyperboloids, which is written as follows [23]:

$$V(p) = \sum_{j=1}^{N} b_j \varphi(p - p_j), \tag{23}$$

where

$$\varphi(p - p_j) = (\|p - p_j\|^2 + C^2)^{\frac{1}{2}}, \qquad j = 1, \ldots, N, \tag{24}$$

where $\|p - p_j\|^2$ is the squared Euclidean distances in $\mathbb{R}$ and C is a non-zero parameter that is determined by the user. The value of this parameter affects the shape; hence it is known as the shape parameter. The function φ is continuously differentiable. The unknown coefficients $\{b_j\}_{j=1}^{N}$ will be determined by solving the following linear equation:

$$V(p_i) = \sum_{j=1}^{N} b_j \varphi(p_i - p_j), \qquad i = 1, \ldots, N. \tag{25}$$

The shape parameter plays an important role in the multi–quadric approximation, and it affects the accuracy and stability of the approximation. There are two kinds of shape parameters: Constant shape parameter and variable shape parameter.

Many people use the constant shape parameter due to its simple analysis in approximation [18, 19]. But multiple results of large set of applications show the advantages of using variable shape parameter; for example, using of variable shape parameter creates more distinct entries of discretization matrix, which reduces the rounding error in calculations [20–22].

In this paper, we use the exponential shape parameter, which is given by Kansa to get more accurate solutions in numerical results [21]:

$$\varphi(p - p_j) = (\|p - p_j\|^2 + C_j^2)^{(\frac{1}{2})}, \qquad j = 1, \ldots, N,$$

$$C_j^2 = C_{\min}^2 \left(\frac{C_{\max}^2}{C_{\min}^2} \right)^{\left(\frac{j-1}{N-1}\right)}, \qquad j = 1, 2, \ldots, N,$$

where $C_{\max}$ and $C_{\min}$ are the selected input parameters; so that the following ratio is in the given range:

$$\frac{C_{\max}}{C_{\min}} \simeq 10 \;\; to \;\; 10^6.$$

3.2 Discretization in time

The fractional derivative in (21) is a modified right Riemann–Liouville derivative which is defined as follows:

$$\frac{\partial^\alpha V(r,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T \frac{V(r,\xi)-V(r,T)}{(\xi-t)^\alpha}d\xi, & 0<\alpha<1,\\ \frac{\partial V(r,t)}{\partial t}, & \alpha=1. \end{cases} \tag{26}$$

When $\alpha = 1$, the model (21) becomes the classical bond pricing model.

Let $t = T - \tau$; for $0 < \alpha < 1$, we have

$$\begin{aligned} \frac{\partial^\alpha V(r,t)}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T \frac{V(r,\xi)-V(r,T)}{(\xi-t)^\alpha}d\xi \\ &= -\frac{1}{\Gamma(1-\alpha)}\frac{d}{d\tau}\int_{T-\tau}^T \frac{V(r,\xi)-V(r,T)}{(\xi-(T-\tau))^\alpha}d\xi \\ &= -\frac{1}{\Gamma(1-\alpha)}\frac{d}{d\tau}\int_0^\tau \frac{V(r,T-\eta)-V(r,T)}{(\tau-\eta)^\alpha}d\eta. \end{aligned} \tag{27}$$

Then (21) can be rewritten as:

$$\begin{aligned} {}_0D_\tau^\alpha U(r,\tau) &= \frac{\Gamma(1+\alpha)}{2}\rho^2\frac{\partial^2 U(r,\tau)}{\partial r^2} - r\frac{(T-\tau)^{(1-\alpha)}}{\Gamma(2-\alpha)}U(r,\tau) \\ &\quad +(\nu(\gamma-r)-\lambda\rho)\frac{\partial U(r,\tau)}{\partial r}, \\ U(r,0) &= 1, \qquad 0<r<R, \\ \frac{\partial U(0,\tau)}{\partial \tau} - \mu(0)\frac{\partial U(0,\tau)}{\partial r} &= 0, \qquad r=0, \quad 0<\tau<T, \\ \frac{\partial U(R,\tau)}{\partial \tau} - \mu(R)\frac{\partial U(R,\tau)}{\partial r} &= -RU(R,\tau), \qquad r=R, \quad 0<\tau<T, \end{aligned} \tag{28}$$

where the fractional derivative is

$${}_0D_\tau^\alpha U(r,\tau) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\tau}\int_0^\tau \frac{U(r,\eta)-U(r,0)}{(\tau-\eta)^\alpha}d\eta, \quad 0<\alpha<1. \tag{29}$$

For $0 < \alpha \leqslant 1$ and $U(r,\tau) \in C^1$, the modified Riemann–Liouville derivative is as follows:

$$\begin{aligned} {}_0D_\tau^\alpha U(r,\tau) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\tau}\int_0^\tau \frac{U(r,\eta)-U(r,0)}{(\tau-\eta)^\alpha}d\eta \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\tau}\int_0^\tau \frac{U(r,\eta)}{(\tau-\eta)^\alpha}d\eta - \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\tau}\int_0^\tau \frac{U(r,0)}{(t-\eta)^\alpha}d\eta \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\tau}\int_0^\tau \frac{U(r,\eta)}{(\tau-\eta)^\alpha}d\eta - \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)}U(r,0) \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^\tau \frac{\partial U(r,\eta)}{\partial \eta}(\tau-\eta)^{-\alpha}d\eta \\ &= {}_0^C D_\tau^\alpha U(r,\tau), \end{aligned} \tag{30}$$

where ${}_0^C D_\tau^\alpha U(r,\tau)$ is a Caputo derivative [1]. In order that discretize the problem for $0 < \alpha < 1$ in time direction, we substitute τ^{n+1} into (30), then we have:

$${}_0D_\tau^\alpha U(r,\tau^{n+1}) = \frac{1}{\Gamma(1-\alpha)}\int_0^{\tau^{n+1}} \frac{\partial U(r,\eta)}{\partial \eta}(\tau^{n+1}-\eta)^{-\alpha}d\eta, \ 0<\alpha\leqslant 1, \tag{31}$$

where $\tau^{n+1} = \tau^n + \delta\tau$, $\tau^0 = 0$, $n = 0, 1, \ldots, N$, $\delta\tau N = T$. Approximation of the first order derivative in (31) by the following finite difference formula

$$\frac{\partial U(r,\tau)}{\partial \tau} \simeq \frac{U(r,\tau^{n+1}) - U(r,\tau^n)}{\partial \tau}, \qquad \tau \in (\tau^n, \tau^{n+1}). \tag{32}$$

By replacing equation (32) into (31), we obtain

$$\begin{aligned}\frac{\partial^\alpha U(r,\tau^{n+1})}{\partial\tau^\alpha} &= \frac{1}{\Gamma(1-\alpha)}\int_0^{\tau^{n+1}}\frac{\partial U(r,\eta)}{\partial\eta}(\tau^{n+1}-\eta)^{-\alpha}d\eta\\ &= \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{n}\int_{\tau^k}^{\tau^{k+1}}\frac{\partial U(r,\eta)}{\partial\eta}(\tau^{n+1}-\eta)^{-\alpha}d\eta\\ &= \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{n}\frac{U^{k+1}-U^k}{\delta\tau}\int_{\tau^k}^{\tau^{k+1}}(\tau^{n+1}-\eta)^{-\alpha}d\eta,\end{aligned} \tag{33}$$

where $U^k = U(r,\tau^k)$, $k = 0, 1, \dots, N$.

Then, the integral in the equation (33) is solved and we have

$$\frac{\partial^\alpha U^{n+1}}{\partial\tau^\alpha} = a_0\left(U^{n+1} - U^n + \sum_{k=1}^{n} b_k(U^{n-k+1} - U^{n-k})\right), \tag{34}$$

where $a_0 = \frac{\delta\tau^{-\alpha}}{\Gamma(2-\alpha)}$, $b_k = (k+1)^{1-\alpha} - (k)^{1-\alpha}$ and $n = 0, 1, \dots, N$.

By replacing equation (34) into equation (14), we obtain

$$\begin{aligned}& a_0\left(U^{n+1} - U^n + \textstyle\sum_{k=1}^{n} b_k(U^{n-k+1} - U^{n-k})\right)\\ &\quad = \frac{\Gamma(1+\alpha)}{2}\rho^2\frac{\partial^2 U^{n+1}}{\partial r^2} - r\frac{(T-(n+1)\delta\tau)^{(1-\alpha)}}{\Gamma(2-\alpha)}U^{n+1}\\ &\qquad +(\nu(\gamma-r)-\lambda\rho)\frac{\partial U^{n+1}}{\partial r},\\ & U^{n+1}(r,0) = 1, \qquad 0<r<R,\\ & U^{n+1}(0,\tau) - \mu(0)\delta\tau U_r^{n+1}(0,\tau) = U^n(0,\tau), \qquad r=0, \quad 0<\tau<T,\\ & U^{n+1}(R,\tau) - \mu(R)\delta\tau U_r^{n+1}(R,\tau) + R\delta\tau U^{n+1}(R,\tau) = U^n(R,\tau),\\ & r=R, \quad 0<\tau<T.\end{aligned} \tag{35}$$

Now, we solve (35) by the MQ approximation method. Therefore, it can be written as follows:

For $n = 0$,

$$\begin{aligned}& a_0U^1 - \frac{\Gamma(1+\alpha)}{2}\rho^2\frac{\partial^2 U^1}{\partial r^2} + r\frac{(T-\delta\tau)^{(1-\alpha)}}{\Gamma(2-\alpha)}U^1 - (\nu(\gamma-r)-\lambda\rho)\frac{\partial U^1}{\partial r} = a_0U^0,\\ & U^1(r,0) = 1, \qquad 0<r<R,\\ & U^1(0,\tau) - \mu(0)\delta\tau U^1(0,\tau) = U^0(0,\tau), \qquad r=0, \quad 0<\tau<T,\\ & U^1(R,\tau) - \mu(R)\delta\tau U^1(R,\tau) + R\delta\tau U^1(R,\tau) = U^0(R,\tau),\\ & r=R, \quad 0<\tau<T.\end{aligned} \tag{36}$$

For $n \geqslant 1$,

$$\begin{aligned}& a_0U^{n+1} - \frac{\Gamma(1+\alpha)}{2}\rho^2\frac{\partial^2 U^{n+1}}{\partial r^2} + r\frac{(T-(n+1)\delta\tau)^{(1-\alpha)}}{\Gamma(2-\alpha)}U^{n+1} - (\nu(\gamma-r)-\lambda\rho)\frac{\partial U^{n+1}}{\partial r}\\ &\quad = a_0U^n - a_0\textstyle\sum_{k=1}^{n} b_k(U^{n-k+1} - U^{n-k}),\\ & U^{n+1}(r,0) = 1, \qquad 0<r<R,\\ & U^{n+1}(0,\tau) - \mu(0)\delta\tau U_r^{n+1}(0,\tau) = U^n(0,\tau), \qquad r=0, \quad 0<\tau<T,\\ & U^{n+1}(R,\tau) - \mu(R)\delta\tau U_r^{n+1}(R,\tau) + R\delta\tau U^{n+1}(R,\tau) = U^n(R,\tau),\\ & r=R, \quad 0<\tau<T.\end{aligned} \tag{37}$$

Now we approximate $U^n(r)$ by the equation (23) as follows:

$$U^n(r) = \sum_{j=1}^{N} b_j^n \varphi(r - r_j),$$

whrere $b_1, b_2, \ldots, b_N$ are unknowns. Then we consider N collocation points to obtain the values of b_j, $j = 1, \ldots, N$ in the interpolant $U^n(r)$ as:

$$U_i^{n+1} = U^{n+1}(r_j) \simeq \sum_{j=1}^{N} b_j^{n+1} \varphi(r_i - r_j).$$

By reconstruction of equation (25) in the matrix form, we obtain

$$[U]^{n+1} = A[b]^{n+1}, \tag{38}$$

where $[V]^{n+1} = [V_1^{n+1}, V_2^{n+1}, \ldots, V_N^{n+1}]^T$, $[b]^{n+1} = [b_1^{n+1}, b_2^{n+1}, \ldots, b_N^{n+1}]^T$, and A is an $N \times N$ matrix

$$A = \begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1j} & \cdots & \varphi_{1N} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \varphi_{i1} & \cdots & \varphi_{ij} & \cdots & \varphi_{iN} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varphi_{N1} & \cdots & \varphi_{Nj} & \cdots & \varphi_{NN} \end{bmatrix}.$$

Now, by considering equations (36) and (37) in a matrix form, we obtain

$$[D]^1 = B[b]^1,$$

where

$$B = \begin{bmatrix} G(\varphi_{11}) & \cdots & G(\varphi_{1j}) & \cdots & G(\varphi_{1N}) \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ G(\varphi_{i1}) & \cdots & G(\varphi_{ij}) & \cdots & G(\varphi_{iN}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G(\varphi_{N1}) & \cdots & G(\varphi_{Nj}) & \cdots & G(\varphi_{NN}) \end{bmatrix},$$

$$G(*) = \begin{cases} (1 - \mu(0)\delta\tau \frac{d}{dr})(*), & i = 1, \\ \quad (a_0 - \frac{\Gamma(1+\alpha)}{2}\rho^2 \frac{d^2}{dr^2} + r\frac{(T-\delta\tau)^{1-\alpha}}{\Gamma(2-\alpha)} & 1 < i < N, \\ \quad -(\nu(\gamma - r) - \lambda\rho)\frac{d}{dr})(*), & \\ (1 - \mu(R)\delta\tau \frac{d}{dr} + R\delta\tau)(*), & i = N, \end{cases}$$

and $[D]^1 = [D_1^1, D_2^1, \ldots, D_N^1]^T$,

$$D_i^1 = \begin{cases} U_1^0, & i = 1, \\ a_0 U_i^0, & 1 < i < N, \\ U_N^0, & i = N. \end{cases}$$

Similarly, equation (37) can be written as:

$$[D]^{n+1} = B[b]^{n+1},$$

where

$$G(*) = \begin{cases} (1-\mu(0)\delta\tau\frac{d}{dr})(*), & i=1, \\ \begin{array}{l}(a_0 - \frac{\Gamma(1+\alpha)}{2}\rho^2\frac{d^2}{dr^2} + r\frac{(T-(n+1)\delta\tau)^{1-\alpha}}{\Gamma(2-\alpha)} \\ -(\nu(\gamma-r)-\lambda\rho)\frac{d}{dr})(*),\end{array} & 1<i<N, \\ (1-\mu(R)\delta\tau\frac{d}{dr} + R\delta\tau)(*), & i=N, \end{cases}$$

and $[D]^{n+1} = [D_1^{n+1}, D_2^{n+1}, \dots, D_N^{n+1}]^T$,

$$D_i^{n+1} = \begin{cases} U_1^n, & i=1, \\ a_0U_i^n - a_0\sum_{k=1}^n b_k(U_i^{n-k+1} - U_i^{n-k}), & 1<i<N, \\ U_N^n, & i=N. \end{cases}$$

4 The stability and convergence analysis

According to section (3.2), we discuss the unconditional stability and convergence of equation (14). We consider the following equation:

$$\frac{\partial^\alpha U}{\partial\tau^\alpha} = a_0\left(\sum_{k=0}^n b_k(U^{n-k+1} - U^{n-k})\right) + \eta_1^n, \tag{39}$$

according to [3], $|\eta_1^n| \leqslant c(\delta\tau)^2$ and

$$\begin{cases} b_k > 0, \qquad k = 0,1,\dots,n, \\ 1 = b_0 > b_1 > \dots > b_n, \qquad b_n \to 0, \qquad n\to\infty, \\ \sum_{k=0}^n (b_k - b_{k+1}) + b_{n+1} = (1-b_1) + \sum_{k=1}^{n-1}(b_k - b_{k+1}) + b_n = 1. \end{cases} \tag{40}$$

Now, we can reconstruct (14) by (34),

$$\begin{aligned} a_0U^{n+1} &- \tfrac{\Gamma(1+\alpha)}{2}\rho^2\tfrac{\partial^2U^{n+1}}{\partial r^2} + r\tfrac{(T-(n+1)\delta\tau)^{1-\alpha}}{\Gamma(2-\alpha)}U^{n+1} - (\nu(\gamma-r)-\lambda\rho)\tfrac{\partial U^{n+1}}{\partial r} \\ &= a_0U^n - a_0\textstyle\sum_{k=1}^n b_k(U^{n-k+1} - U^{n-k}), \end{aligned} \tag{41}$$

the right-hand side of equation (41) can be rewritten as follows:

$$\begin{aligned} &a_0U^n - a_0\textstyle\sum_{k=1}^n b_k(U^{n-k+1} - U^{n-k}) \\ &\quad = a_0(b_0U^n - \textstyle\sum_{k=1}^n b_k(U^{n-k+1} - U^{n-k})) \\ &\quad = a_0(b_0U^n - \textstyle\sum_{k=0}^{n-1} b_{k+1}U^{n-k} + \sum_{k=1}^n b_kU^{n-k}). \end{aligned} \tag{42}$$

We have

$$(u,v) = \int_\Omega (uv)dx,$$

and $\|v\| = (v,v)^{\frac{1}{2}}$ in $\mathcal{L}^2$.

By the inner product of equation (41) in v, we obtain

$$\begin{aligned} & a_0(U^{n+1},v) - \tfrac{\Gamma(1+\alpha)}{2}\rho^2(\tfrac{\partial U^{n+1}}{\partial r},\tfrac{\partial v}{\partial r}) \\ & \quad + r\tfrac{(T-(n+1)\delta\tau)^{1-\alpha}}{\Gamma(2-\alpha)}(U^{n+1},v) - (\nu(\gamma-r)-\lambda\rho)(U^{n+1},\tfrac{\partial v}{\partial r}) \\ & \quad = a_0(b_0(U^n,v) - \textstyle\sum_{k=0}^{n-1} b_{k+1}(U^{n-k},v) \\ & \qquad + \textstyle\sum_{k=1}^{n} b_k(U^{n-k},v)). \end{aligned} \tag{43}$$

Theorem 4.1. *The semidiscrete equation (41) is an unconditional stable for $\delta\tau > 0$ and*

$$\|U^{n+1}\| \leqslant \|U^0\|, \qquad n = 0,1,\ldots,N-1.$$

Proof. Consider equation (43), and let $n=0$; then

$$\begin{aligned} & a_0(U^1,v) - \tfrac{\Gamma(1+\alpha)}{2}\rho^2(\tfrac{\partial U^1}{\partial r},\tfrac{\partial v}{\partial r}) \\ & \quad + r\tfrac{(T-\delta\tau)^{1-\alpha}}{\Gamma(2-\alpha)}(U^1,v) - (\nu(\gamma-r)-\lambda\rho)(U^1,\tfrac{\partial v}{\partial r}) \\ & \quad = a_0 b_0(U^0,v), \end{aligned} \tag{44}$$

by substituting $v = U^1$ in (44) and by using Schwartz inequality, we obtain

$$\|U^1\|^2 \leqslant \|U^1\|\|U^0\| \Rightarrow \|U^1\| \leqslant \|U^0\|.$$

Now, suppose

$$\|U^j\| \leqslant \|U^0\|, \qquad j = 1,2,\ldots,n; \tag{45}$$

we want to prove $\|U^{n+1}\| \leqslant \|U^0\|$. Replacing $v = U^{n+1}$ in (43), implies

$$\begin{aligned} & a_0(U^{n+1},U^{n+1}) - \tfrac{\Gamma(1+\alpha)}{2}\rho^2(\tfrac{\partial U^{n+1}}{\partial r},\tfrac{\partial U^{n+1}}{\partial r}) \\ & \quad + r\tfrac{(T-(n+1)\delta\tau)^{1-\alpha}}{\Gamma(2-\alpha)}(U^{n+1},U^{n+1}) - (\nu(\gamma-r)-\lambda\rho)(U^{n+1},\tfrac{\partial U^{n+1}}{\partial r}) \\ & \quad = a_0(b_0(U^n,U^{n+1}) - \textstyle\sum_{k=0}^{n-1} b_{k+1}(U^{n-k},U^{n+1}) \\ & \qquad + \textstyle\sum_{k=1}^{n} b_k(U^{n-k},U^{n+1})). \end{aligned} \tag{46}$$

The right-hand side of equation (46) can be rewritten as follows:

$$\begin{aligned} & a_0(b_0(U^n,U^{n+1}) - \textstyle\sum_{k=0}^{n-1} b_{k+1}(U^{n-k},U^{n+1}) + \textstyle\sum_{k=1}^{n} b_k(U^{n-k},U^{n+1})) \\ & \quad = a_0((1-b_1)(U^n,U^{n+1}) - \textstyle\sum_{k=1}^{n-1} b_{k+1}(U^{n-k},U^{n+1}) \\ & \qquad + \textstyle\sum_{k=1}^{n} b_k(U^{n-k},U^{n+1})) \\ & \quad = a_0((1-b_1)(U^n,U^{n+1}) + \textstyle\sum_{k=1}^{n-1}(b_k-b_{k+1})(U^{n-k},U^{n+1}) \\ & \qquad + b_n(U^0,U^{n+1})). \end{aligned} \tag{47}$$

By using the Schwartzs inequality, we have

$$\|U^{n+1}\|^2 \leqslant (1-b_1)\|U^n\|\|U^{n+1}\| + \sum_{k=1}^{n-1}(b_k-b_{k+1})\|U^{n-k}\|\|U^{n+1}\| + b_n\|U^0\|\|U^{n+1}\|; \tag{48}$$

therefore, equation (48) leads to

$$\|U^{n+1}\| \leqslant (1-b_1)\|U^n\| + \sum_{k=1}^{n-1}(b_k - b_{k+1})\|U^{n-k}\| + b_n\|U^0\|. \tag{49}$$

According to equation (45),

$$\|U^{n+1}\| \leqslant (1-b_1)\|U^0\| + \sum_{k=1}^{n-1}(b_k - b_{k+1})\|U^0\| + b_n\|U^0\|, \tag{50}$$

by considering equation (40), we obtain

$$\|U^{n+1}\| \leqslant ((1-b_1) + \sum_{k=1}^{n-1}(b_k - b_{k+1}) + b_n)\|U^0\| \leqslant \|U^0\|.$$

$\square$

Theorem 4.2. *Let $(U(r,t^n))_{n=0}^N$ be the exact solution of (14), and let $(U^n)_{n=0}^N$ be the discrete time solution of (14) with initial $U(r,0)=1$; then we have the following error estimates,*

$$\|U(r,t^n) - U^n\| \leqslant \frac{c}{1-\alpha} T^\alpha (\delta\tau)^{2-\alpha}, \quad 0<\alpha<1. \tag{51}$$

Proof. By replacing equation (39) into (14)

$$\begin{aligned} &a_0((U(r,t^{n+1}),v)) - \tfrac{\Gamma(1+\alpha)}{2}\rho^2(\tfrac{\partial U(r,t^{n+1})}{\partial r}, \tfrac{\partial v}{\partial r}) \\ &\quad + r\tfrac{(T-(n+1)\delta\tau)^{1-\alpha}}{\Gamma(2-\alpha)}(U(r,t^{n+1}),v) - (\nu(\gamma - r) - \lambda\rho)(U(r,t^{n+1}), \tfrac{\partial v}{\partial r}) \\ &\quad = a_0\Big(b_0(U(r,t^n),v) - \textstyle\sum_{k=0}^{n-1} b_{k+1}(U(r,t^{n-k}),v) \\ &\qquad + \textstyle\sum_{k=1}^{n} b_k(U(r,t^{n-k}),v)\Big) + (\eta_1^{n+1},v). \end{aligned} \tag{52}$$

By subtracting equation (52) from equation (43), we define $e^{n+1} = U(r,t^{n+1}) - U^{n+1}$; we have

$$\begin{aligned} &a_0(e^{n+1},v) - \tfrac{\Gamma(1+\alpha)}{2}\rho^2(\tfrac{\partial e^{n+1}}{\partial r}, \tfrac{\partial v}{\partial r}) \\ &\quad + r\tfrac{(T-(n+1)\delta\tau)^{1-\alpha}}{\Gamma(2-\alpha)}(e^{n+1},v) - (\nu(\gamma - r) - \lambda\rho)(e^{n+1}, \tfrac{\partial v}{\partial r}) \\ &\quad = a_0\Big((1-b_1)(e^n,v) + \textstyle\sum_{k=1}^{n-1}(b_k - b_{k+1})(e^{n-k},v) \\ &\qquad + b_n(e^0,v)\Big) + (\eta_1^{n+1},v). \end{aligned} \tag{53}$$

By substituting $v = e^{n+1}$ into equation (53), we obtain

$$\begin{aligned} &a_0(e^{n+1},e^{n+1}) - \tfrac{\Gamma(1+\alpha)}{2}\rho^2(\tfrac{\partial e^{n+1}}{\partial r}, \tfrac{\partial e^{n+1}}{\partial r}) \\ &\quad + r\tfrac{(T-(n+1)\delta\tau)^{1-\alpha}}{\Gamma(2-\alpha)}(e^{n+1},e^{n+1}) - (\nu(\gamma - r) - \lambda\rho)(e^{n+1}, \tfrac{\partial e^{n+1}}{\partial r}) \\ &\quad = a_0\Big((1-b_1)(e^n,e^{n+1}) + \textstyle\sum_{k=1}^{n-1}(b_k - b_{k+1})(e^{n-k},e^{n+1}) \\ &\qquad + b_n(e^0,e^{n+1})\Big) + (\eta_1^{n+1},e^{n+1}), \end{aligned} \tag{54}$$

and for $n = 0$, we have

$$\begin{array}{l} a_0(e^1, e^1) - \frac{\Gamma(1+\alpha)}{2}\rho^2(\frac{\partial e^1}{\partial r}, \frac{\partial e^1}{\partial r}) \\ +r\frac{(T-(n+1)\delta\tau)^{1-\alpha}}{\Gamma(2-\alpha)}(e^1, e^1) - (\nu(\gamma - r) - \lambda\rho)(e^1, \frac{\partial e^1}{\partial r}) \\ = a_0((1 - b_1)(e^0, e^1)) + (\eta_1^1, e^1). \end{array} \tag{55}$$

The Schwartz inequality gives

$$\|e^1\|^2 \leqslant \|e^1\|\|e^0\| + \|\eta_1^1\|\|e^1\|; \tag{56}$$

according to (40) and $|\eta_1^n| \leqslant c(\delta\tau)^2$,

$$\|e^1\| \leqslant cb_0^{-1}(\delta\tau)^2.$$

Now, we suppose that inequality $e^n \leqslant cb_{n-1}^{-1}(\delta\tau)^2$ holds for n; we want to prove $\|e^{n+1}\| \leqslant cb_n^{-1}(\delta\tau)^2$.

By using the Schwartz inequality in equation (54), we have

$$\|e^{n+1}\|^2 \leqslant (1-b_1)\|e^n\|\|e^{n+1}\|+\sum_{k=1}^{n-1}(b_k-b_{k+1})\|e^{n-k}\|\|e^{n+1}\|+\|\eta_1^{n+1}\|\|e^{n+1}\|, \tag{57}$$

$$\|e^{n+1}\| \leqslant c(\delta\tau)^2((1 - b_1)b_{n-1}^{-1} + \sum_{k=1}^{n-1}(b_k - b_{k+1})b_{n-k-1}^{-1}) + c(\delta\tau)^2, \tag{58}$$

by applying the induction assumption and $\frac{b_k}{b_{k+1}} < 1$, we obtain

$$\begin{array}{ll} \|e^{n+1}\| & \leqslant c(\delta\tau)^2 b_n^{-1}\left((1 - b_1) + \sum_{k=1}^{n-1}(b_k - b_{k+1})\right) + c(\delta\tau)^2 \\ & = c(\delta\tau)^2 b_n^{-1}\left((1 - b_1) + \sum_{k=1}^{n-1}(b_k - b_{k+1})\right) + cb_n b_n^{-1}(\delta\tau)^2 \\ & = \left(\left((1 - b_1) + \sum_{k=1}^{n-1}(b_k - b_{k+1})\right) + b_n\right) cb_n^{-1}(\delta\tau)^2 \\ & \leqslant cb_n^{-1}(\delta\tau)^2. \end{array} \tag{59}$$

Therefore, the estimation is proved. By using the definition of b_k, we have

$$n^{-\alpha}b_{n-1}^{-1} \leqslant \frac{1}{1-\alpha};$$

consequently, we have

$$\begin{array}{ll} \|U(r, t^n) - U^n\| & \leqslant cb_{n-1}^{-1}(\delta\tau)^2 \leqslant cb_{n-1}^{-1}n^{-\alpha}n^{\alpha}(\delta\tau)^2 \\ & \leqslant \frac{c}{1-\alpha}(n\delta\tau)^{\alpha}(\delta\tau)^2(\delta\tau)^{-\alpha} \leqslant \frac{c}{1-\alpha}T^{\alpha}(\delta\tau)^{2-\alpha}, \end{array}$$

for all n, such that $n\delta\tau \leqslant T$. □

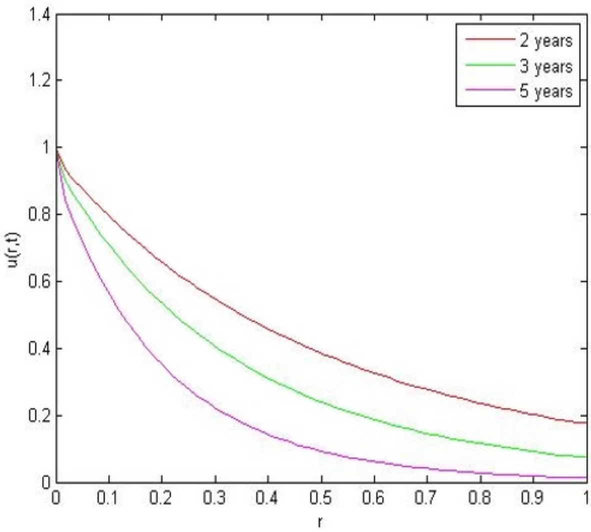

Figure 1: Numerical solution of fractional bond pricing of 5, 7, and 10 years and α=1.

5 Numerical result

In this section, we investigate the efficiency of the presented method. The method is implemented for solving some examples with different parameters.

Example 5.1. Consider equation (21) with the following parameters [27]:

$$\rho = 0.0126, \quad \nu = 0.025, \quad \gamma = 0.15339, \quad C_{\max} = 50, \quad C_{\min} = 1.9.$$

Figure 1 shows the numerical solution, which is obtained by the multiquadric approximation method for the value of 2, 3, and 5 years bonds based on the Vasicek model. It depicts the interest rate from 0 to 1 and the price of bond limits to 0.

Example 5.2. In this example, we consider the actual data of the treasury Bills of the central bank of the Islamic republic of Iran in 2016 and 2017. By using actual data, the parameters of equation (21) are estimated as follows:

$$\rho = 0.0299, \quad \nu = 1.2859, \quad \gamma = 0.2009.$$

Now we plot the treasury Bills figure by considering the other variables and parameters and by using the numerical method for $\alpha = 1, 0.75, 0.5$ at $t = 1$ as follows:

$$r_{\max} = R = 0.25, \quad r_{\min} = 0.18, \quad T = 2,$$

$$C_{\max} = 50, \quad C_{\min} = 1.9.$$

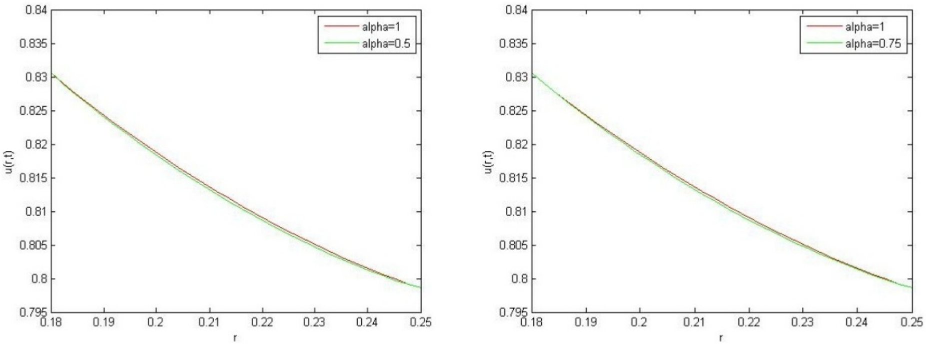

Figure 2: Numerical solution of the bond pricing at $t = 1$.

According to the figure 2, we can find that the price of bonds using the numerical method for different α is limited to the price of the classical bonds. It indicates that the numerical method is efficient.

6 Conclusion and recommendation

The main focus of this paper was to model fractional bond price and to solve it based on the MQ method. In order to reach this goal, first we modelled the bond pricing under vasicek short interest rate model, then we solved this equation by MQ method in which the shape parameter was variable.

The numerical result emphasizes that when the interest rate raises, the price of the bond reduces. Also, by increasing the maturity time of the bonds, the possibility of fluctuating in prices will be higher, due to the interest rate.

Finally, for future research, we suggest to add the jump term to our recommended model to obtain a new model. Also, we can use other interest rate models in this equation and compare the behavior of the models. Other numerical methods can be used for this recommended model.

Bibliography

[1] I. Podlubny, *Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Mathematics in Science and Engineering, 198. Academic Press, Inc., San Diego, CA, 1999.

[2] V. R. Hosseini, W. Chen, and Z. Avazzadeh, *Numerical solution of fractional telegraph equation by using radial basis functions*, Eng. Anal. Bound. Elem, 38 (2014) 31–39.

[3] Y. Lin and C. Xu, *Finite difference/spectral approximations for the time-fractional diffusion equation*, J. Comput. Phys, 225(2) (2007) 1533–1552.

[4] C. Li, Z. Zhao and, Y. Q. Chen, *Numerical approximation of nonlinear fractional differential equations with subdiffusion and superdiffusion*, Comput. Math. Appl, 62(3) (2011) 855–875.

[5] X. Li and C. Xu, *A space-time spectral method for the time fractional diffusion equation*, SIAM J. Numer. Anal, 47(3) (2009) 2108–2131.

[6] F. Liu, P. Zhuang, V. Anh, and I. Turner, *A fractional-order implicit difference approximation for the space–time fractional diffusion equation*, ANZIAM J, 47 (C) (2005) C48–C68.

[7] W. Wyss, *The fractional Black-Scholes equation*, Fract. Calc. Appl. Anal, 3(1) (2000) 51–62.

[8] G. Jumarie, *Derivation and solutions of some fractional Black–Scholes equations in coarse-grained space and time*, Application to Merton's optimal portfolio. Comput. Math. Appl, 59(3) (2010) 1142–1164.

[9] B. Osu and A. Chukwunezu, *On the solution to a fractional Black–Scholes equation for the price of an option*, Int. J. Math. Anal. Appl, 1(3) (2014) 38–42.

[10] M. A. M. Ghandehari and M. Ranjbar, *European option pricing of fractional Black–Scholes model with new Lagrange multipliers*, Comput. Methods Differ. Equ, 2(1) (2014) 1–10.

[11] L. Song and W. Wang, *Solution of the fractional Black–Scholes option pricing model by finite difference method. Abstr. Appl. Anal*, Hindawi Publishing Corporation, (2013).

[12] A. A. Elbeleze, A. Kılıçman, and B. M. Taib, *Homotopy perturbation method for fractional Black–Scholes European option pricing equations using Sumudu transform*, Math. Probl. Eng, 2013.

[13] S. Kumar, A. Yildirim, Y. Khan, H. Jafari, K. Sayevand, and L. Wei, *Analytical solution of fractional Black-Scholes European option pricing equation by using Laplace transform*, J. Fractional Calc. Appl, 2(8) (2012) 1–9.

[14] P. Wilmott, S. Howison and J. Dewynne, *The mathematics of financial derivatives: A student introduction*, Cambridge University Press, 1995.

[15] Y. Yajima, *On Estimation of Long-Memory time series models*, Austral. J. Statist, 27(3) (1985) 303–320.

[16] B. B. Mandelbrot, *When can price be arbitraged efficiently? A limit to the validity of the random walk and martingale models*, Rev. Econ. Stat, 53(3) (1971) 225–236.

[17] H. K. Liu and J. J. Chang, *A closed–form approximation for the fractional Black–Scholes model with transaction costs*, Comput. Math. Appl, 65(11) (2013) 1719–1726.

[18] E. J. Kansa and R. E. Carlson. *Improved accuracy of multiquadric interpolation using variable shape parameters*, Comput. Math. Appl, 24(12) (1992) 99–120.

[19] E. J. Kansa, *Multiquadrics–A scattered data approximation scheme with applications to computational fluid–dynamics.I. surface approximations and partial derivative estimates*, Comput. Math. Appl, 19(8-9) (1990) 127–145.

[20] Y. K. Kwok, *Mathematical models of financial derivatives*, Second edition. Springer Finance. Springer, Berlin, 2008.

[21] G. Jumarie, *Stock exchange fractional dynamics defined as fractional exponential growth driven by (usual) Gaussian white noise. Application to fractional BlackScholes equations*, Insurance: Mathematics and Economics, 42 (2008) 271–287.

[22] Z. C. Deng, J. N. Yu and L. Yang, *An inverse problem arisen in the zero-coupon bond pricing*, Nonlinear Analysis: Real World Applications, 11(3) (2010) 1278–1288

[23] R. Franke, *Scattered data interpolation: tests of some methods*, Math. Comp, 38(157) (1982) 181–200.

[24] C. S. Huang, C. F. Lee, and A. H. D. Cheng, *Error estimate, optimal shape factor, and high precision computation of multiquadric collocation method*, Eng. Anal. Bound. Elem, 31(7) (2007) 614–623.

[25] R. E. Carlson and T. A. Foley, *The parameter R^2 in multiquadric interpolation*, Comput. Math Appl, 21(9) (1991) 29–42.

[26] S. A. Sarra and D. Sturgill, *A random variable shape parameter strategy for radial basis function approximation methods*, Eng. Anal. Bound. Elem, 33(11) (2009) 1239–1245.

[27] J. Xue, *Pricing Callable Bonds*, (Project Report) Uppsala University, 2011.

How to Cite: Sedighe Sharifian[1], Ali R. Soheili[2], Abdolsadeh Neisy[3], *A Numerical solution for the new model of time-fractional bond pricing: Using a multiquadric approximation method*, Journal of Mathematics and Modeling in Finance (JMMF), Vol. 2, No. 1, Pages:107–122, (2022).