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Option valuation in markets with finite liquidity under fractional CEV assets

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Abstract:
The aim of this paper is to numerically price the European double barrier option by calculating the governing fractional Black-Scholes equation in illiquid markets. Incorporating the price impact into the underlying asset dynamic, which means that trading strategies affect the underlying price, we consider markets with finite liquidity. We survey both cases of first-order feedback and full feedback. Asset evolution satisfies a stochastic differential equation with fractional noise, which is more realistic in markets with statistical dependence. Moreover, the Sinc-collocation method is used to price the option. Numerical experiments show that the results highly correspond to our expectation of illiquid markets.

Keywords: Option pricing, Illiquid market, Sinc collocation method, Price impact.
MSC2010 Classification: 91B26, 26A33, 65R20.

1 Introduction

After the financial crisis, market traders realized that a better understanding of the limited liquidity influences on all features of the financial market was needed. One of the origins of such effects is the inclusion of the price impact of option hedging strategies resulting from the relaxation of the assumption of infinite liquidity of the market in underlying assets, which implies that trading affects the price of underlying assets, unlike in Black-Scholes markets. Models that incorporate such an effect unavoidably lead to nonlinear feedback. Inspired by [1] and the references therein, we consider two groups of feedback: First, a hedging strategy that does not consider the feedback effect (first-order feedback with linear governing PDE); second, a hedging strategy that does take into account the feedback effect (full feedback with nonlinear PDE).

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In this paper, we assume that the process of asset price, $\{\mathrm{S}_t, t \geq 0\}$, follows the fractional CEV diffusion model of the form [1]

$$\mathrm{dS}_t = (r - \mathrm{D})\mathrm{S}_t \mathrm{d}t + \sigma(t, \mathrm{S}_t)\mathrm{S}_t \mathrm{dB}_t^{\mathrm{H}}, \tag{1}$$

where $r(t)$ is the interest rate, $\mathrm{D}(t)$ denotes the dividend yield paid to the stock and $\sigma(t, \mathrm{S}_t) = \gamma \mathrm{S}_t^{\rho}$ is the volatility function, in which γ is constant and $\rho \leq 0$ is the elasticity factor. Therefore, when $\mathrm{S} \to \infty$, the local volatility function σ is bounded. Also, $\mathrm{B}_t^{\mathrm{H}}$ denotes the fractional Brownian motion (fBm), stochastic noise, which is a continuous Gaussian process, neither Markov nor semi-martingale and depends on the Hurst parameter. The Hurst parameter $0 < \mathrm{H} \leq 1$ classifies a time series into different groups. If $\mathrm{H} = \frac{1}{2}$, the returns are uncorrelated. For $\mathrm{H} < \frac{1}{2}$, the time series have anti-persistent behavior (short-range memory), and for $\mathrm{H} > \frac{1}{2}$ persistent behavior (long-range memory) is experienced. The Hurst index H describes the raggedness of the stochastic motion, with a higher value leading to a smoother process. When $\mathrm{H} = \frac{1}{2}$, the fBm is exactly the classical Brownian motion with no memory effect. The parameter H allows us to model the statistical dependence of the log returns. Brownian motion increments are independent, while the fBm increments are serially correlated. Therefore, new information has a persisting effect on the process, which implies a certain level of predictability. Thereby, unlike the classical Brownian motion, the historical path of the process matters when its future evolution is forecasting [2].

In our finite liquidity market setup, a price impact term is added to the CEV process (1), i.e.

$$\mathrm{dS}_t = (r - \mathrm{D})\mathrm{S}_t \mathrm{d}t + \gamma \mathrm{S}_t^{\rho+1} \mathrm{dB}_t^{\mathrm{H}} + \lambda(t, \mathrm{S}_t)\mathrm{dF}(t, \mathrm{S}_t), \tag{2}$$

in which $\mathrm{F}(t, \mathrm{S}_t)$ is the number of extra shares traded due to some deterministic hedging strategy and $\lambda(t, \mathrm{S}_t)$ is a function depended to the model of price impact [3]. Applying Itô's formula to $\mathrm{F}(t, \mathrm{S})$ results

$$\mathrm{dF} = \frac{\partial \mathrm{F}}{\partial t}\mathrm{d}t + \frac{\partial \mathrm{F}}{\partial \mathrm{S}}\mathrm{dS} + \frac{1}{2}\frac{\partial^2 \mathrm{F}}{\partial \mathrm{S}^2}(\mathrm{dS})^2 + ...,$$

which substituting into (2), we get leading order

$$(1 - \lambda\frac{\partial \mathrm{F}}{\partial \mathrm{S}})\mathrm{dS}_t = \left[(r - \mathrm{D})\mathrm{S}_t + \lambda\frac{\partial \mathrm{F}}{\partial t}\right]\mathrm{d}t + \gamma \mathrm{S}_t^{\rho+1}\mathrm{dB}_t^{\mathrm{H}} + \frac{\lambda}{2}\frac{\partial^2 \mathrm{F}}{\partial \mathrm{S}^2}(\mathrm{dS}_t)^2. \tag{3}$$

By squaring (3) and using $\left(\mathrm{dB}_t^{\mathrm{H}}\right)^2 \to (dt)^{\alpha}$ $(2\mathrm{H} = \alpha)$, we derive an expression for $(\mathrm{dS}_t)^2$ as $dt \to 0$

$$(\mathrm{dS}_t)^2 = \frac{\gamma^2 \mathrm{S}_t^{2(\rho+1)}}{(1 - \lambda\frac{\partial \mathrm{F}}{\partial \mathrm{S}})^2}(\mathrm{d}t)^{\alpha} + \mathrm{O}\left((\mathrm{d}t)^{\frac{\alpha}{2}+1}\right). \tag{4}$$

Replacing (4) into (3) yields

$$dS_t = \mu(t, S_t)dt + \hat{\sigma}(t, S_t)dB_t^H + \frac{\lambda \hat{\sigma}^2(t, S_t)}{2(1 - \lambda \frac{\partial F}{\partial S})} \frac{\partial^2 F}{\partial S^2} (dt)^\alpha, \tag{5}$$

in which

$$\mu(t, S) := \frac{1}{(1 - \lambda \frac{\partial F}{\partial S})} \left[(r - D)S + \lambda \frac{\partial F}{\partial t} \right], \qquad \hat{\sigma}(t, S) := \frac{\gamma S^{\rho+1}}{(1 - \lambda \frac{\partial F}{\partial S})}.$$

Furthermore, we have (See Eq. (4.11) in [4])

$$(dt)^\alpha = \alpha!(1 - \alpha)! t^{\alpha - 1} dt. \tag{6}$$

Thus, substituting in Eq. (5), we obtain

$$dS_t = \hat{\mu}(t, S_t)dt + \hat{\sigma}(t, S_t)dB_t^H, \tag{7}$$

where

$$\hat{\mu}(t, S) := \mu(t, S) + \frac{\lambda t^{\alpha-1}}{2(1 - \lambda \frac{\partial F}{\partial S})} \alpha!(1 - \alpha)! \hat{\sigma}^2(t, S) \frac{\partial^2 F}{\partial S^2}.$$

Using Maruyama's notation of fractional order [4], we can approximate the fractional CEV diffusion process (7) with

$$dS_t = \hat{\mu}(t, S_t)dt + \hat{\sigma}(t, S_t)\omega(t)(dt)^{\frac{\alpha}{2}}, \tag{8}$$

in which $\omega(t)$ is the companion Gaussian white noise associated with B_t^H, apparently with the mathematical expectation $\mathbb{E}\{\omega(t)\} = 0$ and the variance $\mathbb{E}\{\omega^2(t)\} = 1$ [5].

The rest of the paper is organized as follows: In Section 2, we derive a time-fractional Black-Scholes (BS) equation under the generalized CEV model (8). Section 3 explains the main properties of the Sinc function. In Section 4, a collocation scheme based on the Sinc function is implemented. Some numerical test examples are illustrated in Section 5. Finally, Section 6 contains the main conclusions.

2 Derivation of fractional Black-Scholes equation

In the present study, inspired by [6], we derive the time-fractional BS equation governing the price of the European double barrier option. Suppose that $V(S, t)$ be the price of the European double barrier option, where the underlying asset price follows the generalized CEV process (8).

According to the fractional Taylor's series [4], we have

$$\begin{aligned} dV &= \frac{1}{\alpha!} \frac{\partial^\alpha V}{\partial t^\alpha} (dt)^\alpha + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (dS)^2 \\ &= (1 - \alpha)! t^{\alpha-1} \frac{\partial^\alpha V}{\partial t^\alpha} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (dS)^2. \end{aligned} \tag{9}$$

Also, from Eq. (8), we have

$$
\begin{aligned}
(\mathrm{dS}_t)^2 = \hat{\mu}^2(t,\mathrm{S}_t)(\mathrm{d}t)^2 + \hat{\mu}(t,\mathrm{S}_t)\hat{\sigma}(t,\mathrm{S}_t)\omega(t)(\mathrm{d}t)^{\frac{\alpha}{2}+1} + \hat{\sigma}^2(t,\mathrm{S}_t)\omega^2(t)(\mathrm{d}t)^{\alpha} \\
&\rightarrow \hat{\sigma}^2(t,\mathrm{S}_t)(\mathrm{d}t)^{\alpha} \\
&= \alpha!(1-\alpha)!t^{\alpha-1}\hat{\sigma}^2(t,\mathrm{S}_t)\mathrm{d}t, \qquad (10)
\end{aligned}
$$

where in the last equality we have used Eq. (6).
Hence, by replacing (8) and (10) into Eq. (9), we obtain

$$
\begin{aligned}
\mathrm{dV} = \Big[(1-\alpha)!t^{\alpha-1}\frac{\partial^{\alpha}\mathrm{V}}{\partial t^{\alpha}} + \frac{t^{\alpha-1}}{2}\alpha!(1-\alpha)!\hat{\sigma}^2(t,\mathrm{S}_t)\frac{\partial^2\mathrm{V}}{\partial \mathrm{S}^2} \\
+ \hat{\mu}(t,\mathrm{S}_t)\frac{\partial \mathrm{V}}{\partial \mathrm{S}}\Big]\mathrm{d}t + \hat{\sigma}(t,\mathrm{S}_t)\frac{\partial \mathrm{V}}{\partial \mathrm{S}}\omega(t)(\mathrm{d}t)^{\frac{\alpha}{2}}. \qquad (11)
\end{aligned}
$$

To conclude the fractional BS equation, we assume that market is arbitrage free. Thus, we construct a riskless portfolio $\mathcal{P}(t)$ of option price $\mathrm{V}(\mathrm{S},t)$ and underlying asset price S_t, and also, a dividend process $\mathcal{D}(t)$, as follows

$$\mathcal{P}(t) = \Delta \mathrm{S}_t - \mathrm{V}(\mathrm{S},t), \qquad (12)$$

$$\mathcal{D}(t) = \mathrm{DS}_t \Rightarrow \mathrm{d}\mathcal{D}(t) = \mathrm{DS}_t\mathrm{d}t, \qquad (13)$$

in which Δ denotes shares of underlying asset for hedging the portfolio such that $\mathrm{d}\mathcal{P}(t) = r\mathcal{P}(t)\mathrm{d}t$. Hence

$$r\mathcal{P}(t)\mathrm{d}t = \mathrm{d}\mathcal{P}(t) = \Delta(\mathrm{dS}_t + \mathrm{d}\mathcal{D}(t)) - \mathrm{dV}(\mathrm{S},t). \qquad (14)$$

Thus, by replacing Eqs. (7), (11) and (13) into Eq. (14), we have

$$
\begin{aligned}
r\mathcal{P}(t)\mathrm{d}t = \Big[\Delta(\hat{\mu}(t,\mathrm{S}) + \mathrm{DS}) - (1-\alpha)!t^{\alpha-1}\frac{\partial^{\alpha}\mathrm{V}}{\partial t^{\alpha}} - \hat{\mu}(t,\mathrm{S})\frac{\partial \mathrm{V}}{\partial \mathrm{S}} \\
- \frac{t^{\alpha-1}}{2}\alpha!(1-\alpha)!\hat{\sigma}^2(t,\mathrm{S})\frac{\partial^2\mathrm{V}}{\partial \mathrm{S}^2}\Big]\mathrm{d}t + (\Delta - \frac{\partial \mathrm{V}}{\partial \mathrm{S}})\hat{\sigma}(t,\mathrm{S})\omega(t)(\mathrm{d}t)^{\frac{\alpha}{2}}. \qquad (15)
\end{aligned}
$$

Since the hedging portfolio is risk free, choosing $\Delta = \frac{\partial \mathrm{V}}{\partial \mathrm{S}}$, we remove the coefficient of random term from (15). Thus, from Eqs. (12) and (15), we obtain

$$r(\mathrm{S}\frac{\partial \mathrm{V}}{\partial \mathrm{S}} - \mathrm{V}) = \mathrm{DS}\frac{\partial \mathrm{V}}{\partial \mathrm{S}} - (1-\alpha)!t^{\alpha-1}\frac{\partial^{\alpha}\mathrm{V}}{\partial t^{\alpha}} - \frac{t^{\alpha-1}}{2}\alpha!(1-\alpha)!\hat{\sigma}^2(t,\mathrm{S})\frac{\partial^2\mathrm{V}}{\partial \mathrm{S}^2},$$

and by replacing $\hat{\sigma}(t,\mathrm{S})$, we get

$$\frac{\partial^{\alpha}\mathrm{V}}{\partial t^{\alpha}} + \frac{\gamma^2\alpha!\mathrm{S}^{2(\rho+1)}}{2(1-\lambda\frac{\partial \mathrm{F}}{\partial \mathrm{S}})^2}\frac{\partial^2\mathrm{V}}{\partial \mathrm{S}^2} + \frac{t^{1-\alpha}}{(1-\alpha)!}\left[(r-\mathrm{D})\mathrm{S}\frac{\partial \mathrm{V}}{\partial \mathrm{S}} - r\mathrm{V}\right] = 0, \qquad (16)$$

where $\frac{\partial^{\alpha}\mathrm{V}}{\partial t^{\alpha}}$ denotes the right modified Riemann-Liouville derivative of order $\alpha \in (0,1)$.

The trading strategy F in (16) should be chosen an option delta based on some form of option V^*, i.e., $F = \frac{\partial V^*}{\partial S}$ since $\Delta = \frac{\partial V}{\partial S}$ [3]. This implies that the strategy the hedgers are assumed to follow matters now. A naive strategy is $V^* = V^{BS}$ for the BS value which is not a solution of (16). This leads to the *linear* fractional PDE

$$\frac{\partial^\alpha V}{\partial t^\alpha} + \frac{\gamma^2 \alpha! S^{2(\rho+1)}}{2(1-\lambda\frac{\partial^2 V^{BS}}{\partial S^2})^2}\frac{\partial^2 V}{\partial S^2} + \frac{t^{1-\alpha}}{(1-\alpha)!}\left[(r-D)S\frac{\partial V}{\partial S} - rV\right] = 0. \tag{17}$$

This case is called *first-order feedback*.

Another case is when the hedger is assumed to be aware of the feedback effect and so accordingly would change the hedging strategy. This case, which is called *full feedback*, corresponds to the case $V^* = V$, when the adopted trading strategy has to be found as a part of the problem. This leads to the *nonlinear* fractional PDE

$$\frac{\partial^\alpha V}{\partial t^\alpha} + \frac{\gamma^2 \alpha! S^{2(\rho+1)}}{2(1-\lambda\frac{\partial^2 V}{\partial S^2})^2}\frac{\partial^2 V}{\partial S^2} + \frac{t^{1-\alpha}}{(1-\alpha)!}\left[(r-D)S\frac{\partial V}{\partial S} - rV\right] = 0. \tag{18}$$

The European double barrier knock-out option price $V(S,t)$ is considered with the payoff and boundary conditions as follows:
Call option:

$$\begin{cases} V(S,T) = \max\{S-K,0\}, & S \in (B_l, B_u), \\ V(B_l,t) = \varphi_l(t), & t \in (0,T), \\ V(B_u,t) = \varphi_u(t), & t \in (0,T), \end{cases} \tag{19}$$

Put option:

$$\begin{cases} V(S,T) = \max\{K-S,0\}, & S \in (B_l, B_u), \\ V(B_l,t) = \varphi_l(t), & t \in (0,T), \\ V(B_u,t) = \varphi_u(t), & t \in (0,T), \end{cases} \tag{20}$$

where B_l, B_u are the lower and upper boundary barriers, respectively.
Changing the variable $\tau = T - t$ in the right modified Riemann-Liouville derivative $\frac{\partial^\alpha V}{\partial t^\alpha}$ yields the following Caputo fractional derivative (see Eqs. (13)-(15) in [6])

$$\frac{\partial^\alpha V}{\partial t^\alpha}(S,t) = -{}_0D_\tau^\alpha V(S, T-\tau). \tag{21}$$

Moreover, by changing the variables $x = \ln(S)$ in Eqs. (17)-(18), we obtain the following time-fractional BS equations.

First-order feedback:

$$
\begin{aligned}
{}_0\mathrm{D}_\tau^\alpha u(x,\tau) = & \frac{\gamma^2 \alpha! e^{2\rho x}}{2\phi(e^x, \mathrm{T}-\tau)} (u_{xx}(x,\tau) - u_x(x,\tau)) \\
& + \frac{(\mathrm{T}-\tau)^{1-\alpha}}{(1-\alpha)!} \left[(r(\mathrm{T}-\tau) - \mathrm{D}(\mathrm{T}-\tau)) u_x(x,\tau) - r(\mathrm{T}-\tau) u(x,\tau) \right],
\end{aligned}
\tag{22}
$$

where

$$
\phi(z,y) := \left(1 - \lambda(z,y) \frac{\partial^2 \mathrm{V}^{\mathrm{BS}}}{\partial z^2}(z,y) \right)^2,
$$

Full feedback:

$$
\begin{aligned}
{}_0\mathrm{D}_\tau^\alpha u(x,\tau) = & \frac{\gamma^2 \alpha! e^{2\rho x}}{2} \frac{(u_{xx}(x,\tau) - u_x(x,\tau))}{(1 - e^{-2x}\lambda(e^x, \mathrm{T}-\tau)[u_{xx}(x,\tau) - u_x(x,\tau)])^2} \\
& + \frac{(\mathrm{T}-\tau)^{1-\alpha}}{(1-\alpha)!} \left[(r(\mathrm{T}-\tau) - \mathrm{D}(\mathrm{T}-\tau)) u_x(x,\tau) - r(\mathrm{T}-\tau) u(x,\tau) \right],
\end{aligned}
\tag{23}
$$

where $u(x,\tau) = \mathrm{V}(\mathrm{e}^x, \mathrm{T}-\tau)$ and the operator ${}_0\mathrm{D}_\tau^\alpha[\cdot]$ denotes the Caputo fractional derivative defined as [7]:

$$
{}_0\mathrm{D}_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u_s(x,s) \mathrm{d}s, \quad \alpha \in (0,1), \tag{24}
$$

with the following initial and boundary conditions:
for *Call option:*

$$
\begin{cases}
u(x,0) = \max\{e^x - \mathrm{K}, 0\}, & x \in \Omega, \\
u(b_l,\tau) = \varphi_l(\mathrm{T}-\tau), & \tau \in (0,\mathrm{T}), \\
u(b_u,\tau) = \varphi_u(\mathrm{T}-\tau), & \tau \in (0,\mathrm{T}),
\end{cases}
\tag{25}
$$

for *Put option:*

$$
\begin{cases}
u(x,0) = \max\{\mathrm{K} - e^x, 0\}, & x \in \Omega, \\
u(b_l,\tau) = \varphi_l(\mathrm{T}-\tau), & \tau \in (0,\mathrm{T}), \\
u(b_u,\tau) = \varphi_u(\mathrm{T}-\tau), & \tau \in (0,\mathrm{T}),
\end{cases}
\tag{26}
$$

in which $\Omega := [b_l, b_u]$, $b_d := \ln(\mathrm{B}_l)$ and $b_u := \ln(\mathrm{B}_u)$.

3 Fundamental properties

In this section, the preliminaries of the numerical method are presented. The Sinc function, denoted by $\mathbf{Sinc}(x)$, $x \in \mathbb{R}$, is defined by the following formula [8, 9]

$$
\mathbf{Sinc}(x) = \begin{cases} 1, & x = 0, \\ \frac{\sin(\pi x)}{\pi x}, & x \neq 0. \end{cases}
$$

The Sinc expansion of function $\mathcal{F}(x)$ is formulated by

$$\mathcal{F}(x) \simeq \sum_{i=-n}^{n} \mathcal{F}(i\hat{\Delta})\theta(i,\hat{\Delta};x), \quad x \in \mathbb{R}, \tag{27}$$

in which $\hat{\Delta} > 0$ is a step size and the ith Sinc function is

$$\theta(i,\hat{\Delta};x) := \mathbf{Sinc}\Big(\frac{x}{\hat{\Delta}} - i\Big), \quad i \in \mathbb{Z}. \tag{28}$$

For approximating the function $\mathcal{F}(x)$ on the interval $\Omega := (b_l, b_u)$, we apply the double exponential (DE) transformation function $x = \psi(t)$ on (27), such that

$$\psi(t) := \frac{b_u + b_l}{2} + \frac{b_u - b_l}{2}\tanh\Big(\frac{\pi}{2}\sinh(t)\Big),$$

and

$$\{\psi\}^{-1}(x) = \log\Big(\frac{1}{\pi}\log\Big(\frac{x-b_l}{b_u-x}\Big) + \sqrt{1+\Big\{\frac{1}{\pi}\log(\frac{x-b_l}{b_u-x})\Big\}^2}\Big).$$

The interpolation (27) on the interval Ω is given by

$$\mathcal{F}(x) \simeq \mathcal{F}_n(x) = \sum_{i=-n}^{n} \mathcal{F}\Big(\psi(i\hat{\Delta})\Big)\theta\Big(i,\hat{\Delta};\{\psi\}^{-1}(x)\Big).$$

Theorem 3.1. *([10]) Let n be a positive integer, $\mathcal{F} \in \mathrm{L}_{\boldsymbol{z}}(\psi(\mathbf{B}_\iota))$ such that*

$$\mathbf{B}_\iota = \{s \in \mathbb{C} : |Ims| < \iota\}, \qquad \iota \in (0, \frac{\pi}{2}),$$

and $\hat{\Delta}$ is given by $\hat{\Delta} = \frac{1}{n}\log(2n\iota/\boldsymbol{z})$. Thus, we have

$$\max_{x\in\Omega} |\mathcal{F}(x) - \mathcal{F}_n(x)| \leqslant \zeta \exp\Big(\frac{-\pi\iota n}{\log(2\iota n/\boldsymbol{z})}\Big), \tag{29}$$

where the constant ζ is independent of n.

Remark 3.2. ([11]) Let $\mathcal{C} = \psi^{-1}$ and $\theta_i(x) = \theta(i,\hat{\Delta};\{\psi\}^{-1}(x))$. The k-th derivative of $\theta_i(x)$ with respect to $\mathcal{C}$, evaluated at the point $x_j = \psi(j\Delta)$, are given by

$$\delta[i,j;k] := \hat{\Delta}^k \frac{d^k}{d\mathcal{C}^k}[\theta_i(x))]|_{x=x_j}, \quad k = 0,1,2,\dots,$$

in which the values of the derivative for $k = 0,1,2$ are given by

$$\delta[i,j;0] = [\theta_i(x)]|_{x=x_j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

$$\delta[i,j;1] = \hat{\Delta}\frac{d}{d\mathcal{C}}[\theta_i(x)]|_{x=x_j} = \begin{cases} 0, & i = j, \\ \frac{(-1)^{j-i}}{j-i}, & i \neq j, \end{cases}$$

$$\delta[i,j;2] = \hat{\Delta}^2\frac{d^2}{d\mathcal{C}^2}[\theta_i(x)]|_{x=x_j} = \begin{cases} \frac{-\pi^2}{3}, & i = j, \\ \frac{-2(-1)^{j-i}}{(j-i)^2}, & i \neq j. \end{cases}$$

4 Numerical Discussion

In this section, we propose a numerical scheme based on the Sinc function to find a solution for the time-fractional BS equations (22) and (23). So, first we consider a method for the discretization of the time-fractional derivative.

Let the time mesh points $\tau_m = mh, \quad m = 0, 1, ..., \mathrm{M}$ and $h = \frac{\mathrm{T}}{\mathrm{M}}$. Moreover, suppose that $u^m(x)$ denotes an approximation to $u(x,\tau)$ at the time mesh point $\tau = \tau_m$.

Lemma 4.1. *([12]) Let $u(x,\tau) \in \mathbf{C}(\Omega \times (0,\mathrm{T}))$ and $\alpha \in (0,1)$. Applying the forward finite difference operator, we have*

$$ {}_0\mathrm{D}^{\alpha}_{\tau_m} u(x,\tau_m) \cong \sum_{r=1}^{m} \beta_r \{u(x,\tau_{m-r+1}) - u(x,\tau_{m-r})\} + \mathrm{O}(h^{2-\alpha}), $$

where

$$ \beta_r := \frac{r^{1-\alpha} - (r-1)^{1-\alpha}}{\Gamma(2-\alpha)h^{\alpha}}. $$

By applying Lemma 4.1 to (22) and (23), we obtain the time discretization form of *First-order feedback:*

$$ \begin{aligned} (\beta_1 + \varrho_m)u^m(x) - \vartheta_m u^m_x(x) - \frac{\gamma^2 \alpha! e^{2\rho x}}{2\phi(e^x, \mathrm{T} - \tau_m)}(u^m_{xx}(x) - u^m_x(x)) \\ = \beta_1 u^{m-1}(x) - \sum_{r=2}^{m} \beta_r \{u^{m-r+1}(x) - u^{m-r}(x)\}, \end{aligned} \tag{30} $$

and the time discretization form of *Full feedback:*

$$ \begin{aligned} (\beta_1 + \varrho_m)u^m(x) - \vartheta_m u^m_x(x) - \frac{\gamma^2 \alpha! e^{2\rho x}[u^m_{xx}(x) - u^m_x(x)]}{2\left(1 - e^{-2x}\lambda(e^x, \mathrm{T} - \tau_m)[u^m_{xx}(x) - u^m_x(x)]\right)^2} \\ = \beta_1 u^{m-1}(x) - \sum_{r=2}^{m} \beta_r \{u^{m-r+1}(x) - u^{m-r}(x)\}, \end{aligned} \tag{31} $$

where

$$ \vartheta_m := \frac{(\mathrm{T} - \tau_m)^{1-\alpha}}{(1-\alpha)!}(r(\mathrm{T} - \tau_m) - \mathrm{D}(\mathrm{T} - \tau_m)), \quad \varrho_m := \frac{(\mathrm{T} - \tau_m)^{1-\alpha}}{(1-\alpha)!} r(\mathrm{T} - \tau_m), $$

with the initial and boundary conditions

$$ \text{for } \textit{Call option}: \quad u^0(x) = \max\{e^x - \mathrm{K}, 0\}, \quad u^m(b_l) = u^m(b_u) = 0, \tag{32} $$

$$ \text{for } \textit{Put option}: \quad u^0(x) = \max\{\mathrm{K} - e^x, 0\}, \quad u^m(b_l) = u^m(b_u) = 0. \tag{33} $$

Now, we present numerical approximation based on the Sinc function to solve (30)-(33) as:

$$ u^m(x) = \sum_{i=-n}^{n} c^m_i \theta_i(x) = \mathbf{C}_m \Theta(x)^{\mathrm{T}}, \tag{34} $$

where

$$\mathbf{C}_m = \left[c^m_{-n}, c^m_{-n+1}, ..., c^m_{n-1}, c^m_n\right]^{\mathrm{T}}, \tag{35}$$

$$\Theta(x) = \left[\theta_{-n}(x), \theta_{-n+1}(x), ..., \theta_{n-1}(x), \theta_n(x)\right]^{\mathrm{T}}. \tag{36}$$

Assume that $\delta[i,j;k]$, $k = 0,1,2$ on the interval Ω are defined in Remark 3.2. By evaluating $u^m(x)$ at collocation points x_j, $j = -n, ...n$, we have [12]

$$\mathbf{U}^m = \mathbf{C}_m \Phi_0^{\mathrm{T}}, \tag{37}$$

where

$$\Phi_k := \Big[\delta[i,j;k]\Big]_{\bar{\mathbf{n}}\times\bar{\mathbf{n}}}, \qquad \bar{\mathbf{n}} = 2n+1, \quad k = 0,1,2,$$

and

$$\mathbf{U}^m := \Big[u^m(x_{-n}), u^m(x_{-n+1}), ..., u^m(x_{n-1}), u^m(x_n)\Big]^{\mathrm{T}}.$$

The first order and the second order derivatives of (34) at the collocation points x_j, $j = -n, ...n$, are defined by [12]

$$\mathbf{U}^m_x = \mathbf{C}_m \Psi_1^{\mathrm{T}}, \tag{38}$$

$$\mathbf{U}^m_{xx} = \mathbf{C}_m \Psi_2^{\mathrm{T}}, \tag{39}$$

in which

$$\Psi_1 := -\frac{1}{\hat{\Delta}}\Pi\left[\mathcal{C}'\right]\Phi_1,$$

$$\Psi_2 := \frac{1}{\hat{\Delta}^2}\Pi\left[\left[\mathcal{C}'\right]^2\right]\Phi_2 - \frac{1}{\hat{\Delta}}\Pi\left[\mathcal{C}''\right]\Phi_1.$$

Moreover, assume that $\Pi[v]$ denotes a $\bar{\mathbf{n}}\times\bar{\mathbf{n}}$ diagonal matrix

$$\Pi[v] := \mathrm{diag}\Big[v(x_{-n}), v(x_{-n+1}), ..., v(x_{n-1}), v(x_n)\Big].$$

4.1 First-order feedback

We consider a numerical solution for the time discretization form of *first-order feedback* model (30) as

$$u^m(x) = \sum_{i=-n}^{n} c^m_i \theta_i(x) = \mathbf{C}_m \Theta(x)^{\mathrm{T}}. \tag{40}$$

By substituting (40) in (30) and evaluating at the collocation points x_j, $j = -n, ...n$, and also applying the relations (37)-(39), we obtain a system of *linear* algebraic equations in each time level, as

$$\mathbf{C}_m \mathbf{A}_m = \mathbf{B}_m$$

where

$$\mathbf{A}_m := (\beta_1 + \varrho_m)\Phi_0^{\mathrm{T}} - \vartheta_m \Psi_1^{\mathrm{T}} - \Pi\,[\mathrm{P}_m]\,(\Psi_2^{\mathrm{T}} - \Psi_1^{\mathrm{T}}), \tag{41}$$

$$\mathrm{P}_m^i := \frac{\gamma^2 \alpha! e^{2\rho x_i}}{2\phi(e^{x_i}, \mathrm{T} - \tau_m)}, \qquad i = -n, ..., n, \tag{42}$$

$$\mathbf{B}_m := \left(\beta_1 \mathbf{C}_{m-1} - \sum_{r=2}^{m} \beta_r \{\mathbf{C}_{m-r+1} - \mathbf{C}_{m-r}\}\right)\Theta(x)^{\mathrm{T}}, \tag{43}$$

with the initial condition (Call or Put option) $\mathbf{B}_1 = \left[\beta_1 u^0(x_{-n}), ..., \beta_1 u^0(x_n)\right]^{\mathrm{T}}$.

4.2 Full feedback

Similar to Subsection 4.1, we consider a numerical solution for the time discretization form of *full feedback* model (31) as

$$u^m(x) = \sum_{i=-n}^{n} c_i^m \theta_i(x) = \mathbf{C}_m \Theta(x)^{\mathrm{T}}. \tag{44}$$

By substituting (44) in (31) and evaluating at the collocation points x_j, $j = -n, ...n$, and also applying the relations (37)-(39), we obtain a system of *nonlinear* algebraic equations in each time level, as

$$\mathbf{C}_m \mathbf{A}_m - \frac{\mathbf{F}\mathbf{C}_m[\Psi_2^{\mathrm{T}} - \Psi_1^{\mathrm{T}}]}{2\left(1 - \Pi[\mathrm{P}_m]\mathbf{C}_m[\Psi_2^{\mathrm{T}} - \Psi_1^{\mathrm{T}}]\right)^2} = \mathbf{B}_m,$$

in which

$$\mathbf{A}_m := (\beta_1 + \varrho_m)\Phi_0^{\mathrm{T}} - \vartheta_m \Psi_1^{\mathrm{T}}, \tag{45}$$

$$\mathrm{P}_m^i := e^{-2x_i}\lambda(e^{x_i}, \mathrm{T} - \tau_m), \qquad i = -n, ..., n, \tag{46}$$

$$\mathbf{F} := \mathrm{diag}\left[\gamma^2 \alpha! e^{2\rho x_{-n}}, ..., \gamma^2 \alpha! e^{2\rho x_n}\right], \tag{47}$$

$$\mathbf{B}_m := \left(\beta_1 \mathbf{C}_{m-1} - \sum_{r=2}^{m} \beta_r \{\mathbf{C}_{m-r+1} - \mathbf{C}_{m-r}\}\right)\Theta(x)^{\mathrm{T}}, \tag{48}$$

with the initial condition (Call or Put option) $\mathbf{B}_1 = \left[\beta_1 u^0(x_{-n}), ..., \beta_1 u^0(x_n)\right]^{\mathrm{T}}$.

5 Numerical Experiments

Herein, we implement the described method for the models (22) and (23), with the initial and boundary conditions (25) and (26). Let $r(t) = 0.1$, $\mathrm{D}(t) = 0.02$ (for call option), $\mathrm{D}(t) = 0.2$ (for put option), $\sigma_0 = 0.2$, $\mathrm{S}_0 = 0.5$, $\mathrm{T} = 1$, $\mathrm{B}_d = 0.5$, $\mathrm{B}_u = 2.5$, $\mathrm{K} = 1.5$, $\gamma = \frac{\sigma_0}{\mathrm{S}_0^{\rho}}$, and $\hat{\Delta} = \frac{1}{n}\log(n\pi/6)$.

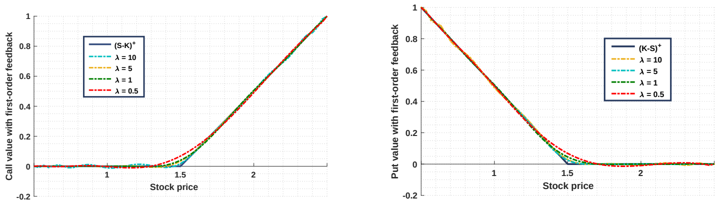

Figure 1: The value of European call and put options with first-order feedback at T = 1 with $\rho = -0.1$ and different values of λ.

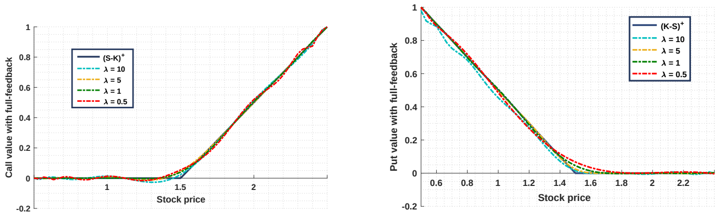

Figure 2: The value of European call and put options with full-feedback at T = 1 with $\rho = -0.1$ and different values of λ.

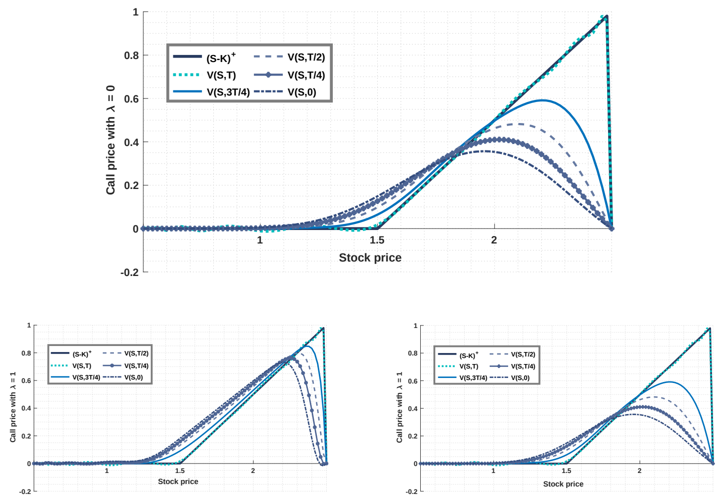

Figure 3: Call option price for different times to expiration date, with $\lambda = 0$ (Up) and $\lambda = 1$ (Down) : [full-feedback (left) and first-order feedback (right)].

Figure 1 and Figure 2 show the values of European barrier call and put options with first-order feedback and full-feedback at T = 1 when S $\in$ $(\mathrm{B}_l, \mathrm{B}_u)$, with

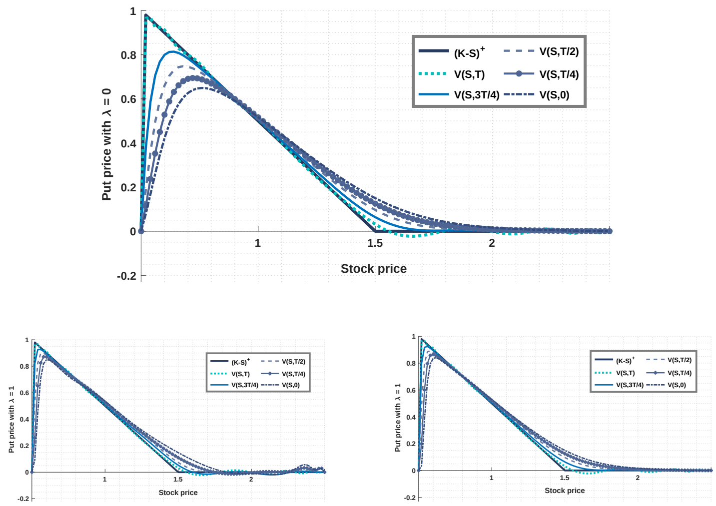

Figure 4: Put option price for different times to expiration date, with $\lambda = 0$ (Up) and $\lambda = 1$ (Down) : [full-feedback (left) and first-order feedback (right)].

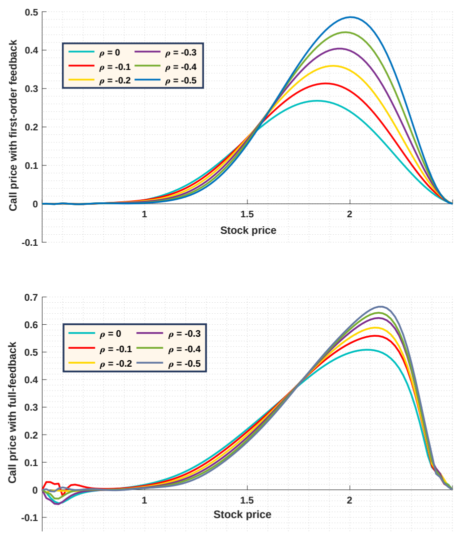

Figure 5: The value of European call option at $\mathrm{T} - t = 1$ with $\alpha = 0.8$, $\lambda = 0.1$ and different values of ρ.

$\rho = -0.1$, $\alpha = 0.9$ and several values of λ, $n = 15$ and $\mathrm{M} = 150$. We see that the results of the numerical scheme sufficiently correspond to the exact payoff, even for large values of λ (increasing the price impact) and also the nonlinear full-feedback

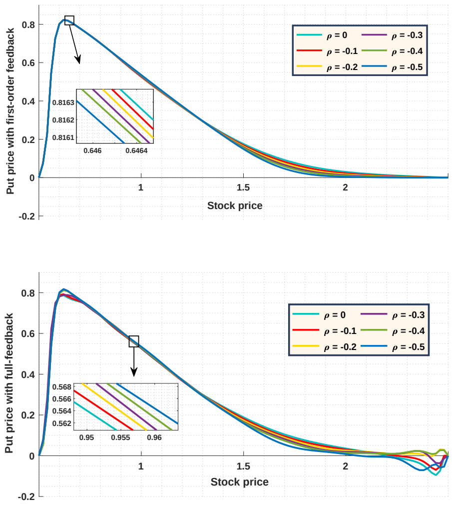

Figure 6: The value of European put option at $\mathrm{T} - t = 1$ with $\alpha = 0.8$, $\lambda = 0.1$ and different values of ρ.

case.

Figure 3 displays the call option price for different times to expiration date, with $\lambda = 0$ (Up, without considering the price impact) and $\lambda = 1$ by full-feedback (Down ; left) and first-order feedback (Down ; right), when $\alpha = 0.95$, $\rho = -0.1$. Figure 4 shows the put option price for different times to expiration date, with $\lambda = 0$ (Up) and $\lambda = 1$ by full-feedback (Down ; left) and first-order feedback (Down ; right), when $\alpha = 0.95$, $\rho = -0.1$. In both cases, the results are surprisingly as expected which shows the efficiency of the numerical scheme in handling the price impact factor.

Figure 5 shows the value of European call option and Figure 6 displays the value of European put option, for first-order feedback and full-feedback at $\mathrm{T} - t = 1$, when $\alpha = 0.8$ and different values of ρ, when $\lambda = 0.1$. It is obvious from the results that we should be cautious about choosing the elasticity factor ρ in our CEV model. The negative value of option which is a sign of friction in the illiquid market, could be avoided in practice by imposing the condition $V \geq 0$, which effectively creates another free boundary on the PDE at $V = 0$.

6 Conclusion

In this research, we consider the fractional CEV equation as the dynamics of asset price, which is a rich model that captures both the volatility smile and the persistent effect of real market data. We have attempted to price the European double-barrier call and put options in the market with finite liquidity. For this purpose, we adopted a collocation approach based on the Sinc functions. The numerical results show that the price impact can be perfectly controlled to make this model a more realistic pricing tool.

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