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Exponential Ornstein-Uhlenbeck model for pricing double barrier options in uncertain environment

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Abstract:
Option pricing is a fundamental issue in financial markets, and barrier options are a popular type of options that can become valuable or worthless when the underlying asset price reaches a predetermined level. A double barrier option consist two barriers, one situated above and the other below the prevailing stock price. This particular option is categorized as path dependent because the return for the holder is influenced by the stock price's breach of the two barriers. The double barrier option contract stipulates three specific payoffs, depending on whether the up-barrier or down-barrier is touched, or if there is no breach of either barrier during the entire duration of the option. In this paper, pricing of the double barrier options when the underlying asset price follows the exponential Ornstein-Uhlenbeck model is investigated, and also pricing formulas for different types of double barrier options (knock-in and knock-out) are derived by α-paths of uncertain differential equations in the uncertain environment.

Keywords: Stock model, Uncertain process, Option Pricing, Double barrier option, Uncertain differential equations.
Classification: MSC2010 or JEL Classifications: 91G20, 91-10, 60K37, 90C70, 91B86, 35A09.

1 Introduction

In the financial markets, options are a remarkable instrument, and their pricing is one of the main subjects in mathematical finance. On the other hand, barrier options and vanilla options are similar, except that the option is knocked in or out when the underlying asset price touches the barrier price before maturity time. Since 1967, barrier options have been traded in the over-the-counter (OTC) market, and nowadays are the most favorite type of exotic options.
Previous techniques in pricing options have been extensively utilized based on the Black-Scholes model [1] and Merton's [18] option pricing theory, in which stochastic

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differential equations (SDEs) were used to model the price process for underlying assets. Merton [18] initially introduced the concept of pricing rational options and subsequently developed to down and out options, then Rich [22] based the pricing of barrier options. After that, researchers have focused on exploring diverse methodologies for valuing such options. For example, Nouri, Abbasi et al. [19], [20] presented an improved Monte Carlo algorithm for pricing different kinds of barrier and double barrier options and [17] applied Lie-algebraic method for finding the value of moving barrier options, and [9] studied the valuation of American double barrier options analytically. [6] studied double barrier option pricing using a regime switching exponential mean reverting process. In 2013, Liu [16] has been suggested that employing stochastic differential equations to characterize the stock price process is inappropriate and leads to a compelling paradox. The empirical phenomena can show this viewpoint by that the peak of distribution of underlying assets is higher than the peak of normal probability distribution and the tails of that is heavier.

However, many empirical studies have indicated that the price of underlying assets does not adhere to the principles of probability and randomness. Instead, financial markets are affected by both randomness and human uncertainty. The degree of investor belief plays a significant role in this, as investors tend to base their decisions on beliefs rather than probabilities. Kahneman [10] indicated that the degrees of beliefs exhibit a wider range of variance compared to frequency.

In 2004, Cont and Tankov [5] used jump-diffusion models as an uncertain source and showed that these models have rich structures for asset pricing. In 2007, Liu [11] within the uncertain measure framework, developed a theory of uncertainty to improve the modeling of uncertain phenomena that dealing with the degree of belief. In 2008, he introduced an uncertain process [12]. Based on it, researchers in [4], [15], [25], and [26] developed methods for solving uncertain differential equations (UDEs). Also, the existence and uniqueness theorem of the solution for UDEs have been demonstrated by Chen and Liu [4]. Moreover, Liu [13] proved the stability of UDEs. Liu [13], several formulas have been derived for option pricing based on the uncertain stock models, in 2009. After that, Pang and Yao [21], Yu [28], Chen [3], Yao [27], and Ji and Zhou [7] attended seriously to uncertain stock pricing models. Besides, Chen [2] introduced a formula for American option pricing in 2011. In 2020, Jia and Chen [8] have presented several interesting findings about the pricing formulas of knock-in barrier options based on an uncertain stock pricing model which has a floating interest rate. In 2020, Rong et al. [23] investigate pricing formulas for American barrier option, and Yang et al. [24] Study approaches for determining the pricing of Asian barrier options in an uncertain environment.

In what follows, some necessary preliminaries are discussed in Section 2. Then, for as much as uncertain space is more accorded to real decision problems, the exponential Ornstein-Uhlenbeck model for stock pricing in uncertain space, is presented in Section 3. Section 4 proves pricing formulas for double barrier (knock-in and knock-out) options for an uncertain stock model. The paper then concluded with a summary in Section 5.

2 Preliminaries

Suppose L be a σ-algebra on a non-empty set Γ(universal set). If $\mathcal{M}$ is a set function $\mathcal{M} : L \to [0,1]$ and it satisfies the following axioms:
1: (Normality axiom) $\mathcal{M}(\Gamma) = 1$;
2: (Subadditivity axiom) For each sequence of events $\{\Theta_j\}$ that can be counted, we have

$$\mathcal{M}(\bigcup_{j=1}^{\infty} \Theta_j) \leq \sum_{j=1}^{\infty} \mathcal{M}(\Theta_j)$$

3: (Duality axiom) $\mathcal{M}(\Theta) + \mathcal{M}(\Theta^c) = 1$ for every event Θ;
Then, (Γ, L) is a measurable space, and the triplet $(\Gamma, L, \mathcal{M})$ is an uncertain space.
4: (Monotonicity axiom) $\mathcal{M}(\Theta_1) \leq \mathcal{M}(\Theta_2)$ every where $\Theta_1 \subseteq \Theta_2$;

Definition 2.1. [13]. The set function $\mathcal{M}$, which satisfies the above axioms is called an uncertain measure.

5: (Product Axiom) [13]. Let the triple $(\Gamma_k, L_k, \mathcal{M}_k)$, where $\Gamma = \Gamma_1 \times \Gamma_2 \times ...\Gamma_k$ and $L = L_1 \times L_2 \times ...L_k$ be uncertainty spaces for $k = 1, 2, ..., n$, the product uncertain measure $\mathcal{M}$ is an uncertain measure on the σ -algebra satisfying

$$\mathcal{M}(\prod_{k=1}^{\infty} \Theta_k) \leq \bigwedge_{k=1}^{\infty} \mathcal{M}_k(\Theta_k)$$

where Θ_k, for $k = 1, 2, ..., n$ are arbitrary chosen events from L_k, respectively.

Definition 2.2. [14]. The uncertainty distribution for an uncertain variable such as η is defined by function $\Psi : \mathbb{R} \to [0,1]$ that $\Psi(x) = \mathcal{M}\{\eta \leq x\}$.

Definition 2.3. Following uncertainty distribution is called normal

$$\Psi(x) = \Big(1 + exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\Big)^{-1}, \qquad x \in \mathbb{R}, \tag{1}$$

If η be an uncertain variable, in this case $\sigma > 0$ and e are real numbers and it is shown by $\mathcal{N}(e, \sigma)$. The normal uncertaintly distribution can be called standard, if e be equal to 0 and σ be equal to 1.
The inverse uncertaintly distribution of η denoted by $\Psi^{-1}(\alpha)$, $\alpha \in (0, 1)$ and the expected value of an uncertain variable η is defined as

$$E[\eta] = \int_0^1 \Psi^{-1}(\alpha) d\alpha, \tag{2}$$

Definition 2.4. [12] following UDE (uncertain differential equation)

$$dX_t = h(t, X_t)dt + k(t, X_t)dC_t, \tag{3}$$

has an α-path X_t^α $(0 < \alpha < 1)$, if it solves the bellow corresponding ODE

$$dX_t^\alpha = h(t, X_t^\alpha)dt + |k(t, X_t^\alpha)|\Psi^{-1}(\alpha)dt, \tag{4}$$

where $\Psi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution, i.e.,

$$\Psi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} ln \frac{\alpha}{1-\alpha}, \tag{5}$$

Definition 2.5. [13] Liu process is an uncertain process C_t which have bellow properties:

- $C_0 = 0$,
- C_t has independent and stationary increments,
- almost all sample paths are Lipschitz continuous,
- all increments $C_{s+t} - C_s$ are normal uncertain variables with expected value 0 and variance t^2.

Theorem 2.6. *[25] Let X_t be the solution of the UDE eq.(12) and X_t^α be the solution and α-path of ODE eq.(4). Then*

$$\mathcal{M}\{X_t \le X_t^\alpha, \quad \forall t \in [0, T]\} = \alpha,$$
$$\mathcal{M}\{X_t > X_t^\alpha, \quad \forall t \in [0, T]\} = 1 - \alpha,$$

Theorem 2.7. *[27] Assume that $\eta_1, \eta_2, ..., \eta_m, ..., \eta_n$ are independent uncertain variables and $\Psi_1, \Psi_2, ..., \Psi_m, ..., \Psi_n$ be regular uncertainty distributions of these variables, respectively. if the function $f(x_1, x_2, ..., x_m, x_{m+1}, ..., x_n)$ is strictly increasing function with respect to $x_1, x_2, ..., x_m$ and strictly decreasing function with respect to $x_{m+1}, x_{m+2}..., x_n$, then the uncertain process $\eta = f(\eta_1, ..., \eta_m, ..., \eta_n)$ has an inverse uncertainty distribution*

$$\Psi^{-1}(\alpha) = f(\Psi_1^{-1}(\alpha), ..., \Psi_m^{-1}(\alpha), ..., \Psi_{m+1}^{-1}(1-\alpha), ..., \Psi_n^{-1}(1-\alpha))$$

where $\Psi^{-1}(\alpha) = X_t^\alpha(\alpha - path\ \ of\ X_t)$

3 Uncertain stock model for barrier option pricing

Assum that the stock price S_t follows the exponential Ornstein-Uhlenbeck model:

$$\begin{cases} dS_t = \mu(1 - \lambda ln S_t)S_t dt + \sigma S_t dC_t \\ dP_t = rP_t dt \end{cases} \tag{6}$$

where P_t is the bond price, $\lambda > 0$ is constant, positive constants r, μ, σ are the risk-less interest rate, log-drift and log-diffusion respectively, and C_t represents a Liu process.

Theorem 3.1. *Assume that the stock price follows*

$$dS_t = \mu(1 - \lambda lnS_t)S_t dt + \sigma S_t dC_t. \tag{7}$$

where S_t is the stock price at the moment t. Then we obtain an α-path for S_t as

$$S_t^\alpha = exp[exp(-\mu\lambda t)lnS_0 + (1 - exp(-\mu\lambda t))(\frac{1}{\lambda} + \frac{\sigma\sqrt{3}}{\mu\lambda\pi}ln\frac{\alpha}{1-\alpha})] \tag{8}$$

Proof. According to Definition[2.4], we have

$$dS_t^\alpha = \mu(1 - \lambda lnS_t^\alpha)S_t^\alpha dt + \sigma S_t^\alpha \frac{\sqrt{3}}{\pi}ln\frac{\alpha}{1-\alpha}dt$$

so

$$\frac{dS_t^\alpha}{S_t^\alpha} = \mu(1 - \lambda lnS_t^\alpha)dt + \sigma\frac{\sqrt{3}}{\pi}ln\frac{\alpha}{1-\alpha}dt$$

and

$$dlnS_t^\alpha = (\mu + \frac{\sigma\sqrt{3}}{\pi}ln\frac{\alpha}{1-\alpha})dt - \mu\lambda lnS_t^\alpha dt$$

By solving the above differential equation we have

$$lnS_t^\alpha = (\frac{\mu + \frac{\sigma\sqrt{3}}{\pi}ln\frac{\alpha}{1-\alpha}}{\mu\lambda})(1 - exp(-\mu\lambda t)) + exp(-\mu\lambda t)lnS_0$$

Then, the α-path for S_t as

$$\begin{aligned} S_t^\alpha &= S_0^{exp(-\mu\lambda t)}exp[(1 - exp(-\mu\lambda t))(\frac{1}{\lambda} + \frac{\sigma\sqrt{3}}{\mu\lambda\pi}ln\frac{\alpha}{1-\alpha})] \\ &= exp[exp(-\mu\lambda t)lnS_0 + (1 - exp(-\mu\lambda t))(\frac{1}{\lambda} + \frac{\sigma\sqrt{3}}{\mu\lambda\pi}ln\frac{\alpha}{1-\alpha})] \end{aligned}$$

□

4 double barrier options

In double barrier options, one barrier is above the current stock price and the other is below it. Since the payoff of the option depends on the behavior of the stock price process due to these two obstacles, it is considered as path-dependent option. The payoff of the double barrier options during the life of the option finds three different types, depending on whether it hits the upper barrier or lower barrier or no touched the barrier level. If the payoff of the option decreases after hitting the barrier, the barrier is called worthless (the amount of decrease may be time-dependent), and if the payoff increases, the barrier is called valuable. One of the characteristics of the barrier is that it may be used throughout the life of the option or a part of the life of the option. In this section, we have presented the formula for pricing double barrier options, which asset price follows Eq [10].

4.1 knock-in options

One kind of barrier option is a knock-in option which contract that only comes in existence when the underlying asset crosses a certain price level. This means that traders can buy or sell this type of option only at the moment, and after that, the price reaches a particular prespecified level. If the knock-in price level has touched at any time before the maturity date of the options contract, the payoff of the option is converted into a vanilla option, and the knock-in barrier option expires worthless. In this section, we have presented the formula for pricing European knock-in options, in which asset price follows Eq [10].

Pricing formula for double knock-in call option

Consider a double barrier option which the lower barrier level is B_L, the upper barrier level is B_U, the exercise price is K, and the expiration time is T. If before the maturity T, the underlying asset price S_t hits the lower or upper barrier level and exceeds them, then this call option will become into existence, and its payoff will be $max(S_t - K, 0)$ on the maturity date. Now we assign $\eta^+ = max(\eta, 0)$ and apply an indicator function

$$I_B(\eta) = \begin{cases} 1, & S_t < B_L \ or \ S_t > B_U \\ 0, & B_L < \eta < B_U \end{cases}$$

Hence, the payoff on the maturity time is written as;

$$payoff = (S_T - K)^+(I_B(S_t)) \tag{9}$$

By taking into account the discount rate on the initial date, the discounted expectation of payoff is

$$B_{dki} = e^{-rT}(S_T - K)^+(I_B(S_t)) \tag{10}$$

and a price of this kind of double barrier options is

$$f^c_{dki} = E[B_{dki}] = E[e^{-rT}(S_T - K)^+(I_B(S_t))] \tag{11}$$

Theorem 4.1. *Consider a double knock-in call option for stock pricing model that underlying uncertain Eq. [10] has a lower barrier level B_L, upper barrier level B_U,* *exercise price K, and the expiration time T. Then the price of this option is defined by*

$$f^c_{dki} = e^{-rT}[\int_0^{\theta_0}(S^\alpha_T - K)^+ d\alpha + \int_{\theta_1}^1 (S^\alpha_T - K)^+ d\alpha] \tag{12}$$

where

$$\theta_0 = (1 + exp[\frac{\mu\pi}{\sigma\sqrt{3}}(\frac{\lambda(exp(-\mu\lambda t)lnS_0 - ln(B_L))}{1 - exp(-\mu\lambda t)} + 1)])^{-1} \tag{13}$$

and

$$\theta_1 = (1 + exp[\frac{\mu\pi}{\sigma\sqrt{3}}(\frac{\lambda(exp(-\mu\lambda t)lnS_0 - ln(B_U))}{1 - exp(-\mu\lambda t)} + 1)])^{-1} \tag{14}$$

and

$$S^\alpha_t = exp[exp(-\mu\lambda t)lnS_0 + (1 - exp(-\mu\lambda t))(\frac{1}{\lambda} + \frac{\sigma\sqrt{3}}{\mu\lambda\pi}ln\frac{\alpha}{1-\alpha})] \tag{15}$$

Proof. Note that

$$I_B(S^\alpha_t) = 1 \tag{16}$$

if and only if

$$S^\alpha_t < B_L \quad or \quad S^\alpha_t > B_U \tag{17}$$

In addition

$$\begin{aligned} &exp[exp(-\mu\lambda t)lnS_0 + (1 - exp(-\mu\lambda t))(\frac{1}{\lambda} + \frac{\sigma\sqrt{3}}{\mu\lambda\pi}ln\frac{\alpha}{1-\alpha})] < B_L \\ \Rightarrow &exp(-\mu\lambda t)lnS_0 + (1 - exp(-\mu\lambda t))(\frac{1}{\lambda} + \frac{\sigma\sqrt{3}}{\mu\lambda\pi}ln\frac{\alpha}{1-\alpha}) < ln(B_L) \\ \Rightarrow &\frac{\mu\pi}{\sigma\sqrt{3}}[\frac{\lambda(exp(-\mu\lambda t)lnS_0 - ln(B_L))}{1 - exp(-\mu\lambda t)} + 1] < ln\frac{1-\alpha}{\alpha} \end{aligned}$$

By taking

$$M = \frac{\mu\pi}{\sigma\sqrt{3}}[\frac{\lambda(exp(-\mu\lambda t)lnS_0 - ln(B_L))}{1 - exp(-\mu\lambda t)} + 1] \tag{18}$$

we have

$$e^M < \frac{1-\alpha}{\alpha} \tag{19}$$

then

$$\alpha < \frac{1}{1 + e^M} = \theta_0 \tag{20}$$

On the other hand

$$\begin{aligned} &exp[exp(-\mu\lambda t)lnS_0 + (1 - exp(-\mu\lambda t))(\frac{1}{\lambda} + \frac{\sigma\sqrt{3}}{\mu\lambda\pi}ln\frac{\alpha}{1-\alpha})] > B_U \\ \Rightarrow &exp(-\mu\lambda t)lnS_0 + (1 - exp(-\mu\lambda t))(\frac{1}{\lambda} + \frac{\sigma\sqrt{3}}{\mu\lambda\pi}ln\frac{\alpha}{1-\alpha}) > ln(B_U) \\ \Rightarrow &\frac{\mu\pi}{\sigma\sqrt{3}}[\frac{\lambda(exp(-\mu\lambda t)lnS_0 - ln(B_L))}{1 - exp(-\mu\lambda t)} + 1] > ln\frac{1-\alpha}{\alpha} \end{aligned}$$

By taking

$$N = \frac{\mu\pi}{\sigma\sqrt{3}}[\frac{\lambda(exp(-\mu\lambda t)lnS_0 - ln(B_U))}{1 - exp(-\mu\lambda t)} + 1] \tag{21}$$

we have

$$e^N > \frac{1-\alpha}{\alpha} \tag{22}$$

then

$$\alpha > \frac{1}{1+e^N} = \theta_1 \tag{23}$$

□

Example 4.2. Assume the initial stock price $S_0 = 4$, risk-less interest rate $r = 0.03$, lower barrier level $B_L = 2$, upper barrier level $B_U = 6$, strike price $K = 8$, time to maturity $T = 20$, log-diffusion $\sigma = 0.05$, log-drift $\mu = 0.01$ and parameter $\lambda = 0.6$. Then the price of double knock-in call option is 1.1532.

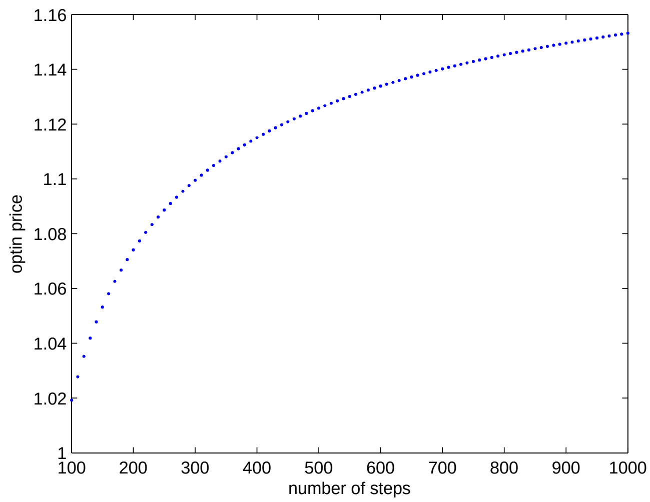

Figure 1: The barrier option price f^c_{dki} with respect to different step N in Example 4.2.

Pricing formula for double knock-in put option

Consider a double barrier option which the lower barrier level is B_L, the upper barrier level is B_U, the exercise price is K, and the expiration time is T. If before the maturity T, the underlying asset price S_t hits the lower or upper barrier level and exceeds them, then this call option will become into existence, and its payoff will be $max(K - S_t, 0)$ on the maturity date. Now we assign $\eta^+ = max(\eta, 0)$ and apply an indicator function

$$I_B(\eta) = \begin{cases} 1, & S_t < B_L \ or \ S_t > B_U \\ 0, & B_L < \eta < B_U \end{cases}$$

Hence, the payoff on the maturity time is written as;

$$payoff = (K - S_T)^+(I_B(S_t)) \tag{24}$$

By taking into account the discount rate on the initial date, the discounted expectation of payoff is

$$B_{dki} = e^{-rT}(K - S_T)^+(I_B(S_t)) \tag{25}$$

and a price of this kind of double barrier options is

$$f^p_{dki} = E[B_{dki}] = E[e^{-rT}(K - S_T)^+(I_B(S_t))] \tag{26}$$

Theorem 4.3. *Consider a double knock-in put option for stock pricing model that underlying uncertain Eq. [10] has a lower barrier level B_L, upper barrier level B_U, exercise price K, and the expiration time T. Then the price of this option is defined by*

$$f^p_{dki} = e^{-rT}[\int_0^{\theta_0}(K - S^\alpha_T)^+ d\alpha + \int_{\theta_1}^1 (K - S^\alpha_T)^+ d\alpha] \tag{27}$$

where

$$\theta_0 = (1 + exp[\frac{\mu\pi}{\sigma\sqrt{3}}(\frac{\lambda(exp(-\mu\lambda t)lnS_0 - ln(B_L))}{1 - exp(-\mu\lambda t)} + 1)])^{-1} \tag{28}$$

and

$$\theta_1 = (1 + exp[\frac{\mu\pi}{\sigma\sqrt{3}}(\frac{\lambda(exp(-\mu\lambda t)lnS_0 - ln(B_U))}{1 - exp(-\mu\lambda t)} + 1)])^{-1} \tag{29}$$

and

$$S^\alpha_t = exp[exp(-\mu\lambda t)lnS_0 + (1 - exp(-\mu\lambda t))(\frac{1}{\lambda} + \frac{\sigma\sqrt{3}}{\mu\lambda\pi}ln\frac{\alpha}{1 - \alpha})] \tag{30}$$

Proof. Similar to theorem 4.1 it will be proofed. □

Example 4.4. Assume the initial stock price $S_0 = 6$, risk-less interest rate $r = 0.03$, lower barrier level $B_L = 5$, upper barrier level $B_U = 15$, strike price $K = 8$, time to maturity $T = 20$, log-diffusion $\sigma = 0.05$, log-drift $\mu = 0.04$ and parameter $\lambda = 0.6$. Then the price of double knock-in put option is 1.1347.

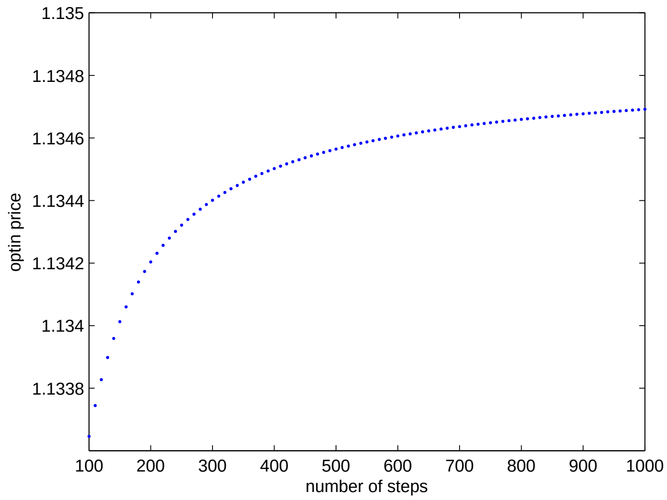

Figure 2: The barrier option price f^{p}_{dki} with respect to different step N in Example 4.4.

4.2 knock-out options

Knock-out option is a kind of barrier option that if the underlying asset price does not exceed a specified barrier level during the life of the option, then it has a payoff, that is if the price of the underlying asset before the maturity date T crosses the barrier level, the payoff becomes zero. In this section, we have presented the formula for pricing European knock-out option, which asset price follows Eq [10].

Pricing formula for double knock-out call option

Consider a double barrier option which the lower barrier level is B_L, the upper barrier level is B_U, the exercise price is K, and the expiration time is T. If before the maturity T, the spot price S_t always be between the lower barrier level B_L and upper barrier level B_U, then this call option will become into existence, and its payoff will be $max(S_t - K, 0)$ on the maturity date. Now we assign $\eta^+ = max(\eta, 0)$ and apply an indicator function

$$I_B(\eta) = \begin{cases} 1, & B_L < \eta < B_U \\ 0, & \eta < B_L \ or \ \eta > B_U \end{cases}$$

Hence, the payoff on the maturity time is written as;

$$payoff = (S_T - K)^+ (I_B(S_t)) \tag{31}$$

By taking into account the discount rate on the initial date, the discounted expectation of payoff is

$$B_{dko} = e^{-rT}(S_T - K)^+ (I_B(S_t)) \tag{32}$$

and a price of this kind of double barrier options is

$$f^c_{dko} = E[B_{dko}] = E[e^{-rT}(S_T - K)^+ (I_B(S_t))] \tag{33}$$

Theorem 4.5. *Consider a double knock-out call option for stock pricing model that underlying uncertain Eq. [10] has a lower barrier level B_L, upper barrier level B_U, exercise price K, and the expiration time T. Then the price of this option is defined by*

$$f^c_{dko} = e^{-rT} \int_{\theta_0}^{\theta_1} (S^\alpha_T - K)^+ d\alpha \tag{34}$$

where

$$\theta_0 = (1 + exp[\frac{\mu\pi}{\sigma\sqrt{3}}(\frac{\lambda(exp(-\mu\lambda t)lnS_0 - ln(B_L))}{1 - exp(-\mu\lambda t)} + 1)])^{-1} \tag{35}$$

and

$$\theta_1 = (1 + exp[\frac{\mu\pi}{\sigma\sqrt{3}}(\frac{\lambda(exp(-\mu\lambda t)lnS_0 - ln(B_U))}{1 - exp(-\mu\lambda t)} + 1)])^{-1} \tag{36}$$

and

$$S^\alpha_t = exp[exp(-\mu\lambda t)lnS_0 + (1 - exp(-\mu\lambda t))(\frac{1}{\lambda} + \frac{\sigma\sqrt{3}}{\mu\lambda\pi} ln\frac{\alpha}{1-\alpha})] \tag{37}$$

Proof. Note that

$$I_B(S^\alpha_t) = 1 \tag{38}$$

if and only if

$$B_L < S^\alpha_t < B_U \tag{39}$$

In addition

$$B_L < exp[exp(-\mu\lambda t)lnS_0 + (1 - exp(-\mu\lambda t))(\frac{1}{\lambda} + \frac{\sigma\sqrt{3}}{\mu\lambda\pi} ln\frac{\alpha}{1-\alpha})] < B_U$$

$$\Rightarrow ln(B_L) < exp(-\mu\lambda t)lnS_0 + (1 - exp(-\mu\lambda t))(\frac{1}{\lambda} + \frac{\sigma\sqrt{3}}{\mu\lambda\pi} ln\frac{\alpha}{1-\alpha}) < ln(B_U)$$

$$\Rightarrow \frac{\mu\pi}{\sigma\sqrt{3}}[\frac{\lambda(exp(-\mu\lambda t)lnS_0 - ln(B_U))}{1 - exp(-\mu\lambda t)} + 1] < ln\frac{1-\alpha}{\alpha} < \frac{\mu\pi}{\sigma\sqrt{3}}[\frac{\lambda(exp(-\mu\lambda t)lnS_0 - ln(B_L))}{1 - exp(-\mu\lambda t)} + 1]$$

By taking

$$N = \frac{\mu\pi}{\sigma\sqrt{3}}[\frac{\lambda(exp(-\mu\lambda t)lnS_0 - ln(B_U))}{1 - exp(-\mu\lambda t)} + 1] \tag{40}$$

and

$$M = \frac{\mu\pi}{\sigma\sqrt{3}}[\frac{\lambda(exp(-\mu\lambda t)lnS_0 - ln(B_L))}{1 - exp(-\mu\lambda t)} + 1] \tag{41}$$

we have

$$e^N < \frac{1-\alpha}{\alpha} < e^M \tag{42}$$

then

$$\theta_0 = \frac{1}{1+e^M} < \alpha < \frac{1}{1+e^N} = \theta_1 \tag{43}$$

□

Example 4.6. Assume the initial stock price $S_0 = 3$, risk-less interest rate $r = 0.03$, lower barrier level $B_L = 2$, upper barrier level $B_U = 10$, strike price $K = 5$, time to maturity $T = 15$, log-diffusion $\sigma = 0.05$, log-drift $\mu = 0.04$ and parameter $\lambda = 0.6$. Then the price of knock-out call option is 0.3222.

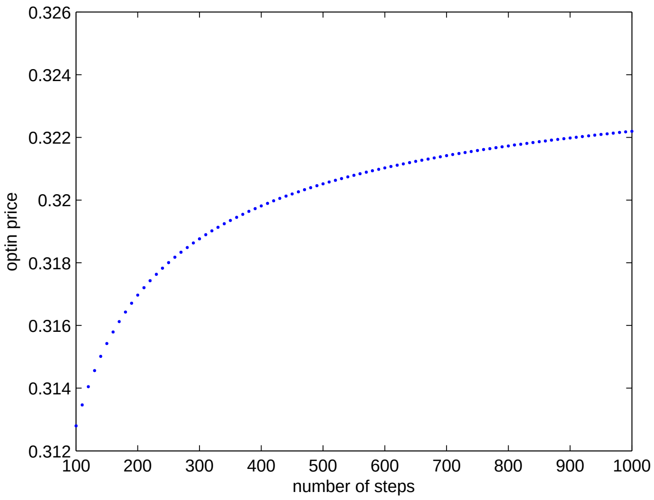

Figure 3: The double barrier option price f^c_{dko} with respect to different step N in Example 4.6.

Pricing formula for double knock-out put option

Consider a double barrier option which the lower barrier level is B_L, the upper barrier level is B_U, the exercise price is K, and the expiration time is T. If before the maturity T, the spot price S_t always be between the lower barrier level B_L and upper barrier level B_U, then this call option will become into existence, and its payoff will be $max(K - S_t, 0)$ on the maturity date. Now we assign $\eta^+ = max(\eta, 0)$ and apply an indicator function

$$I_B(\eta) = \begin{cases} 1, & B_L < \eta < B_U \\ 0, & \eta < B_L \ or \ \eta > B_U \end{cases}$$

Hence, the payoff on the maturity time is written as;

$$payoff = (K - S_T)^+(I_B(S_t)) \tag{44}$$

By taking into account the discount rate on the initial date, the discounted expectation of payoff is

$$B_{dko} = e^{-rT}(K - S_T)^+(I_B(S_t)) \tag{45}$$

and a price of this kind of double barrier options is

$$f^p_{dko} = E[B_{dko}] = E[e^{-rT}(K - S_T)^+(I_B(S_t))] \tag{46}$$

Theorem 4.7. *Consider a double knock-out put option for stock pricing model that underlying uncertain Eq. [10] has a lower barrier level B_L, upper barrier level B_U, exercise price K, and the expiration time T. Then the price of this option is defined by*

$$f^p_{dko} = e^{-rT}\int_{\theta_0}^{\theta_1}(K - S^\alpha_T)^+ d\alpha \tag{47}$$

where

$$\theta_0 = (1 + exp[\frac{\mu\pi}{\sigma\sqrt{3}}(\frac{\lambda(exp(-\mu\lambda t)lnS_0 - ln(B_L))}{1 - exp(-\mu\lambda t)} + 1)])^{-1} \tag{48}$$

and

$$\theta_1 = (1 + exp[\frac{\mu\pi}{\sigma\sqrt{3}}(\frac{\lambda(exp(-\mu\lambda t)lnS_0 - ln(B_U))}{1 - exp(-\mu\lambda t)} + 1)])^{-1} \tag{49}$$

and

$$S^\alpha_t = exp[exp(-\mu\lambda t)lnS_0 + (1 - exp(-\mu\lambda t))(\frac{1}{\lambda} + \frac{\sigma\sqrt{3}}{\mu\lambda\pi}ln\frac{\alpha}{1-\alpha})] \tag{50}$$

Proof. Similar to theorem 4.5 it will be proofed. □

Example 4.8. Assume the initial stock price $S_0 = 10$, risk-less interest rate $r = 0.03$, lower barrier level $B_L = 5$, upper barrier level $B_U = 15$, strike price $K = 8$, time to maturity $T = 10$, log-diffusion $\sigma = 0.05$, log-drift $\mu = 0.04$ and parameter $\lambda = 0.6$. Then the price of double knock-out put option is 0.3215.

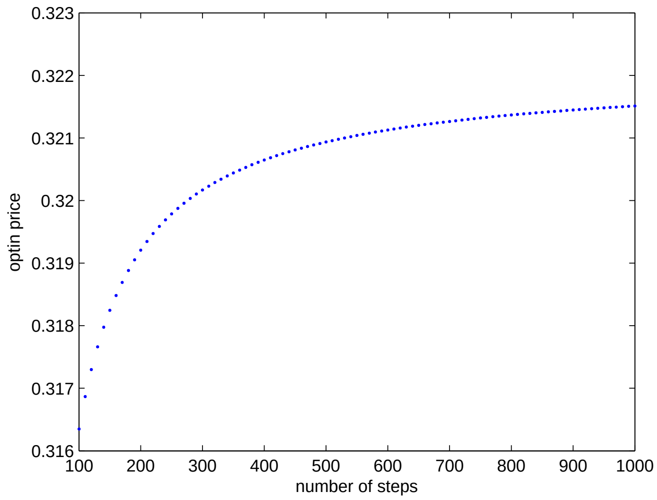

Figure 4: The barrier option price f^p_{dko} with respect to different step N in Example 4.8.

5 Conclusion

Since the probability space and randomness aren't sufficient space for the simulation of investor decisions, many researchers propose Liu's uncertain space for using in such cases. In this paper, we have presented an uncertain process to verify the double barrier option pricing formula. Formulas for pricing knocked-in options and knocked-out options are arrived by α-paths of UDEs in the uncertain environment. In addition, some numerical examples are illustrated for the pricing of double barrier options with the presented model. Further research may use Power and Digital options, or other types of exotic options on this uncertain stock pricing model with similar conditions and may consider multi-asset options in the uncertain environment and derive formulas for option pricing.

Data availability and conflict of interest statement

This work is of theoretical nature and has not analyzed or generated any datasets.
The authors have no conflicts of interest to declare in relation to this article.
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Author contribution statement

B. A and K. N. contributed in designing the model and computational framework, organizing the research and performing numerical simulations and reviewing the results and writing of the manuscript.

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