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A Copula-based estimator for the Sharpe Ratio of a two-asset portfolio

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Abstract: Performance measures are essential for evaluating portfolio performance in the risk management and fund industries, with the Sharpe ratio being a widely adopted risk-adjusted metric. This ratio compares the excess expected return to its standard deviation, enabling investors to assess the returns of risk-taking activities against risk-free options. Its popularity stems from its ease of calculation and straightforward interpretation. However, the actual Sharpe ratio value is often unavailable and must be estimated empirically based on the assumption of normality in asset returns. In practice, financial assets typically exhibit non-normal distributions and nonlinear dependencies, which can compromise the accuracy of Sharpe ratio estimates when normality is assumed. This paper challenges the normality assumption, aiming to enhance the accuracy of Sharpe ratio estimates. We investigate the impact of dependency on the Sharpe ratio of a two-asset portfolio using copulas. Theoretical findings, along with extensive simulations, demonstrate the effectiveness of the proposed copula-based approach compared to the classic Sharpe ratio.

Keywords: Copula, Dependence, Portfolio, Risk-adjusted measure, Sharpe ratio

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1. Introduction

The Sharpe ratio is one of the most popular tools for comparing the performance of financial assets. Sharpe introduced this measure in 1966 Sharpe (1966) and later revised his definition in 1994 Sharpe (1994). The Sharpe ratio is defined as the ratio of the excess expected return to its standard deviation, allowing an investor to compare the profits associated with risk-taking activities to risk-free activities. It is recognized as a reliable measure of risk management. One reason for the popularity of the Sharpe ratio is that it can be easily obtained, and its interpretation is relatively simple. The ratio and its properties have been extensively studied in the literature. For a complete discussion about the Sharpe ratio, we refer to the recent book by Pav Pav (2021) and the many references therein.

In a good investment, both the choice of assets and the selection of a diverse portfolio of assets are important. A key factor in portfolio diversity is the structure of the dependency between the returns of assets in the portfolio. When the returns of assets follow a normal distribution, the dependency is fully described by the linear correlation. That is, the variance of portfolios depends only on the variance of each asset's return portfolio and the linear correlation between them. But in real-world problems, there is considerable evidence of deviations from the normal distribution assumption in asset returns Das and Uppal (2004); Hartmann *et al.* (2004); Harvey and Siddique (1999). Such deviations may result in an incorrect performance evaluation when applying the Sharpe ratio. To solve this problem, various solutions have been proposed in the literature.

A common approach to model the dependency structure of asset returns is the use of time series models; see, e.g., Brooks *et al.* (2005); Capitani (2012); Harvey and Siddique (1999). In financial literature, an alternative approach to addressing the non-normal dependence structure is based on copula theory Cherubini *et al.* (2004); Embrechts *et al.* (2002); Fantazzini (2008); He and Gong (2009). Copulas are effective tools for capturing different types of dependence structures Nelsen (2006). In Mousavi *et al.* (2024), copulas were used for the first time to enhance the accuracy of the maximum Sharpe ratio estimation.

In this paper, we consider a more general case and examine the effect of dependency on various properties of the Sharpe ratio in a two-asset portfolio by using copulas. This study focuses on the static case and emphasizes the copula representation of dependence among random variables. The paper is organized as follows: In Section 2, we define the copula-based Sharpe ratio. Section 2 also provides several examples of the value of the proposed Sharpe ratio under different copula structures. Section 3 is devoted to analyzing the properties of the copula-based Sharpe ratio. In order to compare the value of the empirical Sharpe ratio with that of the copula-based one, a simulation study was carried out in Section 4. In

Section 5, an application to real data is presented. Section 6 concludes.

2. Copula-based Sharpe ratio of a two-asset portfolio

Let R_A and R_B denote the returns of two assets A and B, respectively. Further, let R_f be the return of a benchmark investment strategy. Define continuous random variables X and Y as $X = R_A - R_f$ and $Y = R_B - R_f$. Consider univariate marginal distribution functions of X and Y as $F(x) = P(X \leq x)$ and $G(y) = P(Y \leq y)$ for $x, y \in \mathbb{R}$ with the joint distribution function $H(x,y) = P(X \leq x, Y \leq y)$. Let $T = wX + (1-w)Y$, be a portfolio with dependent components X and Y, where $0 < w < 1$ is the weight of X and $(1-w)$ is the weight of Y. The Sharpe ratio of T is given by

$$\mathrm{SR}_T = \frac{w\mu_X + (1-w)\mu_Y}{\sqrt{w^2\sigma_X^2 + (1-w)^2\sigma_Y^2 + 2w(1-w)\sigma_{X,Y}}}, \tag{2.1}$$

where $\mu_X = E(X)$, $\mu_Y = E(Y)$, $\sigma_X^2 = \mathrm{var}(X)$, $\sigma_Y^2 = \mathrm{var}(Y)$ and $\sigma_{X,Y} = \mathrm{cov}(X,Y)$. Let $SR_X = \mu_X/\sigma_X$ and $SR_Y = \mu_Y/\sigma_Y$ be the Sharpe ratios of X and Y. In formula (2.1), μ_X, μ_Y, σ_X^2 and σ_Y^2 are calculated from the marginal distributions and $\sigma_{X,Y}$ is associated to the joint distribution function of X and Y. Following the *Sklar's Theorem* Nelsen (2006), there exists a unique copula C such that

$$H(x,y) = C(F(x), G(y)), \quad x, y \in \mathbb{R}, \tag{2.2}$$

wherer $C(u,v) = P(U \leq u, V \leq v)$ is the joint distribution of the pair $(U,V) = (F(X), G(Y))$ whose margins are uniform on [0,1]. The copula C characterizes the dependence between the pair (X,Y) Nelsen (2006). By using Hoeffding's identity Hoeffding (1994) and transformations $u = F(x)$ and $v = G(y)$, we have

$$\begin{aligned} \sigma_{X,Y} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [H(x,y) - F(x)G(y)]dxdy \\ &= \int_0^1\int_0^1 [C(u,v) - uv]dF^{-1}(u)dG^{-1}(v). \end{aligned} \tag{2.3}$$

Alternatively, $\sigma_{X,Y}$ can be calculated by

$$\begin{aligned} \sigma_{X,Y} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xydH(x,y) - \mu_X\mu_Y \\ &= \int_0^1\int_0^1 F^{-1}(u)G^{-1}(v)dC(u,v) - \mu_X\mu_Y. \end{aligned} \tag{2.4}$$

When the joint distribution function of (X,Y) is non-normal, we can model it by selecting suitable parametric forms for F, G, and C in (2.2). For example, F

might be normal with the parameters μ and σ^2 and Y might be a gamma random variable with parameters α and β and C might be taken from a parametric family of copulas. Popular choices of copulas are described in Joe (2014); Nelsen (2006). The main advantage of this approach is that distributions F, G, and C in (2.2) can be chosen independently of one another. Copulas are a powerful tool because they can capture various features of financial data, such as asymmetry, non-linear dependence, or tail dependence.

3. Examples

In the following examples, the Sharpe ratio is calculated for some bivariate distributions. Let Π denote the copula of independent random variables, i.e., $\Pi(u,v) = uv$ for all $(u,v) \in [0,1]^2$, and let M and W denote the Fréchet-Hoeffding upper and lower bound copulas, respectively, which, for any copula C, satisfy: $\max(u+v-1,0) = W(u,v) \leq C(u,v) \leq M(u,v) = \min(u,v)$ for every $(u,v) \in [0,1]^2$. We recall that M (W) is the copula of perfect positive (negative) dependence random variables Nelsen (2006). In the following examples, we compute the Sharpe ratio of a portfolio with two perfect dependence assets, having exponentially distributed returns.

Example 3.1. *Let X and Y be two exponential random variables with the means $\dfrac{1}{\lambda_1}$ and $\dfrac{1}{\lambda_2}$, wherer $\lambda_1 > \lambda_2$. If the copula of X and Y is M, then*

$$\begin{aligned}
\sigma_{X,Y} &= \frac{1}{\lambda_1\lambda_2}\int_0^1\int_0^1 \ln(1-u)\ln(1-v)dM(u,v) - \frac{1}{\lambda_1\lambda_2} \\
&= \frac{1}{\lambda_1\lambda_2}\int_0^1 (\ln(1-u))^2 du - \frac{1}{\lambda_1\lambda_2} \\
&= \frac{1}{\lambda_1\lambda_2}.
\end{aligned} \tag{3.5}$$

The Sharpe ratio of the portfolio $T = wX + (1-w)Y$ is then

$$SR_T = \frac{\lambda_1 + (\lambda_2-\lambda_1)w}{|\lambda_1 + (\lambda_2-\lambda_1)w|} = \begin{cases} +1 & if \;\; w > \frac{\lambda_1}{\lambda_1-\lambda_2} \\ 0 & if \;\; w = \frac{\lambda_1}{\lambda_1-\lambda_2} \\ -1 & if \;\; w < \frac{\lambda_1}{\lambda_1-\lambda_2}. \end{cases}$$

If the copula of X and Y is W, then

$$\begin{aligned}\sigma_{X,Y} &= \frac{1}{\lambda_1\lambda_2}\int_0^1\int_0^1 \ln(1-u)\ln(1-v)dW(u,v) - \frac{1}{\lambda_1\lambda_2} \\ &= \frac{1}{\lambda_1\lambda_2}\int_0^1 \ln(1-u)\ln(u)du - \frac{1}{\lambda_1\lambda_2} \\ &= (1-\frac{\pi^2}{6})\frac{1}{\lambda_1\lambda_2}. \end{aligned} \tag{3.6}$$

The Sharpe ratio of the portfolio T is given by

$$SR_T = \frac{\lambda_1 + (\lambda_2 - \lambda_1)w}{\sqrt{(1-w)^2\lambda_2^2 + w^2\lambda_1^2 + (\frac{\pi^2}{3} - 2)w(1-w)}}.$$

The maximum value of SR_T is $\frac{6}{\sqrt{36-3\pi^2}} = 2.3734$, which happens at $w^* = \frac{\lambda_1}{\lambda_1+\lambda_2}$.

Example 3.2. *Let X and Y be two exponential random variables with the means $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$, respectively. Suppose also that we choose to model the dependence by the Farlie-Gumbel-Morgenstern or FGM copula*

$$C_\theta(u,v) = uv[1+\theta(1-u)(1-v)], \;\; u,v \in [0,1],$$

where $-1 \le \theta \le 1$. Then by using (2.3) we have $\sigma_{X,Y} = \dfrac{\theta}{4\lambda_1\lambda_2}$ and the Sharpe ratio of the portfolio T with the exponential returns and FGM copula structure is given by

$$SR_T = \frac{\lambda_1 + (\lambda_2 - \lambda_1)w}{\sqrt{\lambda_2^2 w^2 + \lambda_1^2(1-w)^2 + \frac{w(1-w)\theta\lambda_1\lambda_2}{2}}}. \tag{3.7}$$

As a function of the weight w, the maximum value of SR_T is given by $SR^* = 2\sqrt{\frac{2}{4+\theta}}$, which happens at $w^* = \frac{\lambda_1}{\lambda_1+\lambda_2}$. We note that SR^* is decreasing in θ and for $\theta \in [-1,1]$, $SR^* \in [1.264, 1.633]$. Since θ is the dependency parameter, the value of the Sharpe ratio decreases as the dependence between components of the portfolio increases. For the case of independence, i.e., $\theta = 0$, we have $SR_T^* = \sqrt{2}$.

We note that in a copula-based Sharpe ratio, $E(T) = E(wX + (1-w)Y)$ does not depend on the parameter θ of the copula. However, returns may have distributions whose variance is a function of their mean, as the following example shows.

Example 3.3. *Consider (X_1, X_2) distributed as Marshall–Olkin bivariate exponential distribution with the survival function Marshall and Olkin (1967)*

$$\overline{H}(x,y) = e^{-(\lambda_1 x + \lambda_2 y + \lambda_{12}\max(x,y))}, \quad x,y > 0,$$

where $\lambda_1, \lambda_2, \lambda_{12}$ are positive parameters. The survival copula associated with H is given by $\widehat{C}(u,v) = \min(u^{1-\alpha}, v^{1-\beta})$, where $\alpha = \frac{\lambda_{12}}{\lambda_1+\lambda_{12}}$ and $\beta = \frac{\lambda_{12}}{\lambda_2+\lambda_{12}}$; see, Nelsen (2006). The marginal distribution functions are exponential with $E(X_i) = \frac{1}{\lambda_i+\lambda_{12}}$, $var(X_i) = \frac{1}{(\lambda_i+\lambda_{12})^2}$, $i = 1, 2$ and

$$\sigma_{X_1,X_2} = \frac{\lambda_{12}}{(\lambda_1+\lambda_{12})(\lambda_2+\lambda_{12})(\lambda_1+\lambda_2+\lambda_{12})}.$$

The Sharpe ratio of the two assets portfolio $T = wX_1 + (1-w)X_2$ is given by

$$SR_T = \frac{\lambda_1+\lambda_{12}+(\lambda_2-\lambda_1)w}{\sqrt{(\lambda_2+\lambda_{12})^2w^2+(\lambda_1+\lambda_{12})^2(1-w)^2+\frac{2w(1-w)\lambda_{12}(\lambda_1+\lambda_{12})(\lambda_2+\lambda_{12})}{\lambda_1+\lambda_2+\lambda_{12}}}}.$$

The maximum value of SR_T is given by

$$SR_T^* = \sqrt{\frac{2(\lambda_1+\lambda_2+\lambda_{12})}{\lambda_1+\lambda_2+2\lambda_{12}}},$$

which happens at the point $w^* = \frac{\lambda_1+\lambda_{12}}{\lambda_1+\lambda_2+2\lambda_{12}}$. Note that SR^* is decreasing in the dependence parameter λ_{12}. For the independent case, i.e., $\lambda_{12} = 0$, $SR^* = \sqrt{2}$, which is the maximum value of SR^*, and $SR^* \to 1$, as $\lambda_{12} \to \infty$.

From the ranges of possible values of its numerator and denominator, the Sharpe ratio can theoretically reach any value. Sharpe ratio can theoretically be infinitely large. The following example provides the Sharpe ratio of a two-asset portfolio with the Pareto distributed returns.

Example 3.4. *Let (X, Y) be distributed as a Pareto distribution with the joint survival function $\overline{H}(x,y) = (x+y+1)^{-\theta}$, $x, y > 0$ and $\theta > 0$. The survival copula associated with H is given by $\widehat{C}(u,v) = (u^{-\frac{1}{\theta}} + v^{-\frac{1}{\theta}} - 1)^{-\theta}$, which is a member of Clayton's family of copulas Nelsen (2006). Simple calculation gives $E(X) = E(Y) = \frac{1}{\theta-1}$ and $var(X) = var(Y) = \frac{\theta}{(\theta-1)^2(\theta-2)}$ and $cov(X,Y) = \frac{1}{(\theta-1)^2(\theta-2)}$. The Sharpe ratio of the portfolio $T = wX + (1-w)Y$ is then*

$$SR_T = \frac{\theta\sqrt{\theta-2}}{\sqrt{\theta-2w(1-w)(\theta-1)}}.$$

The maximum value of SR_T is given by $SR_T^* = \theta\sqrt{\frac{2\theta-4}{\theta+1}}$, which happens at $w = \frac{1}{2}$. We note that $SR^* \to \infty$, as $\theta \to \infty$.

4. Dependency properties of copula-based Sharpe ratio

Let $T_{C_\theta} = wX + (1-w)Y$ be a two-asset portfolio whose components (X, Y) have the one-parameter copula structure C_θ. In this section, we discuss some properties of the copula-based Sharpe ratio $SR(T_{C_\theta}) = \frac{\mu(T_{C_\theta})}{\sigma(T_{C_\theta})}$, where $\mu(T_{C_\theta}) = E(T_{C_\theta})$ and $\sigma(T_{C_\theta}) = \sqrt{var(T_{C_\theta})}$. First note that if the variance of returns does not depend on their expected values, then $SR(T_{C_\theta})$ is decreasing (increasing) in θ, as $\sigma(T_{C_\theta})$ is increasing (decreasing) in θ. The following result compares two portfolios with the common marginal distributions and different dependence structures.

Proposition 4.1. *For $i = 1, 2$, let (X_i, Y_i) have the copula C_{θ_i}, $i = 1, 2$, and $E(X_1) = E(X_2)$, $var(X_1) = var(X_2)$. Then $SR(T_{C_{\theta_1}}) \geq SR(T_{C_{\theta_2}})$, whenever C_θ is a positively ordered family of copulas; that is $C_{\theta_1}(u,v) \leq C_{\theta_2}(u,v)$, for all $u, v \in [0,1]$ and $\theta_1 \leq \theta_2$.*

Proof. If $C_{\theta_1}(u,v) \leq C_{\theta_2}(u,v)$, for all $u, v \in [0,1]$, then by (2.1)

$$\int_0^1 \int_0^1 [C_{\theta_1}(u,v) - C_{\theta_2}(u,v)] dF^{-1}(u) dG^{-1}(v) \leq 0.$$

Therefore, $var(T_{C_{\theta_1}}) \leq var(T_{C_{\theta_2}})$ and $SR(T_{C_{\theta_1}}) \geq SR(T_{C_{\theta_2}})$. □

The following result provides a lower and upper bound for Sharpe ratio of a two-asset portfolio with dependent returns.

Proposition 4.2. *Let $T_C = wX + (1-w)Y$ be a two-asset portfolio whose components (X, Y) have the copula structure C. Then*

$$SR(T_M) \leq SR(T_C) \leq SR(T_W),$$

where, M and W are the Fréchet-Hoeffding upper and lower bound copulas,

Proof. The three ratios SR_W, SR_C, and SR_M have a common numerator $\mathrm{E}(wX + (1-w)Y)$. The result follows by using Proposition 1 and the fact that for every copula C, we have $W(u,v) \leq C(u,v) \leq M(u,v)$, for all $u, v \in [0,1]$; see, e.g, Nelsen (2006). Therefore;

$$\mathrm{var}(T_W) \leq \mathrm{var}(T_C) \leq \mathrm{var}(T_M).$$

□

Example 4.3. *Let $T_C = wX + (1-w)Y$ be a portfolio with exponentially distributed returns and the associated copula C. Then, from Example 3.1 and using Proposition 4.2, we have*

$$\frac{\lambda_1 + (\lambda_2 - \lambda_1)w}{|\lambda_1 + (\lambda_2 - \lambda_1)w|} \leq SR(T_C) \leq \frac{\lambda_1 + (\lambda_2 - \lambda_1)w}{\sqrt{(1-w)^2\lambda_2^2 + w^2\lambda_1^2 + (\frac{\pi^2}{3} - 2)w(1-w)}}.$$

A copula C is said to be positive quadrant dependence (PQD) if for all $(u,v) \in [0,1]^2$, $C(u,v) \geq uv$ and negative quadrant dependence (NQD) if $C(u,v) \leq uv$ Nelsen (2006). The following result compares the Sharpe ratio of a portfolio with dependent returns with the Sharpe ratio of a portfolio consisting of independent returns.

Proposition 4.4. *Let $T_C = wX + (1-w)Y$ be a two-asset portfolio whose components (X,Y) have a copula structure C. If C is PQD, then $SR(T_C) \leq SR(T_\Pi)$. If C is NQD then $SR(T_\Pi) \leq SR(T_C)$.*

Proof. If C is PQD, then $C(u,v) \geq uv$ and $\sigma_{X,Y} \geq 0$ so, $SR(T_C) \leq SR(T_\Pi)$. Similarly, if C is NQD, then $\sigma_{X,Y} \leq 0$ so, $SR(T_\Pi) \leq SR(T_C)$. □

Remark 4.5. *The above result shows that, when the returns of a two-asset portfolio are positively or negatively dependent, but considered independent, the Sharpe ratio is more or less estimated.*

Example 4.6. *Let $T_C = wX + (1-w)Y$ be a portfolio with exponentially distributed returns and the associated PQD copula C. Then, from Example 3.2 and using Proposition 4.4, we have*

$$SR(T_C) \leq \frac{\lambda_1 + (\lambda_2 - \lambda_1)w}{\lambda_1\sqrt{2w^2 - 2w + 1}}.$$

5. Simulation study

To compare the value of the empirical Sharpe ratio with the value of the copula-based one, a simulation study was carried out according to a factorial design involving four factors that affect the estimation process:

(1) sample size: $n \in \{50, 200, 500\}$;

(2) degree of dependence in terms of Kendall's tau $\tau(\theta) = 4\int_0^1\int_0^1 C_\theta(u,v)dC_\theta(u,v) - 1$ at $\tau \in \{-0.3, 0.3, 0.8\}$;

(3) dependence structure, represented by the copula C:

- Clayton (an asymmetric copula with the lower tail dependence)

$$C_\theta(u,v) = \{\max(u^{-\theta} + v^{-\theta} - 1, 0)\}^{-\frac{1}{\theta}}, \quad \theta \in [-1,\infty) - \{0\};$$

- Frank (a symmetric copula)

$$C_\theta(u,v) = -\frac{1}{\theta}\ln\left(1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1}\right), \quad \theta \in (-\infty,\infty) - \{0\};$$

- Gumbel (an asymmetric copula with the upper tail dependence)

$$C_\theta(u,v) = \exp\{-[(-\ln(u))^\theta + (-\ln(v))^\theta]^{\frac{1}{\theta}}\}, \quad \theta \in [1,\infty);$$

(4) type of marginal distributions:

- symmetric (normal distribution)
- heavy-tail symmetric (logistic distribution)
- skew (lognormal distribution).

For each combination of factors, 1000 random samples were generated and the Sharpe ratio values were computed. The results are shown in Tables 1–8. Tables are separated based on three types of marginal distributions. The tables give the copula structure (Model), dependency level (Kendall's τ), the true value of the Sharpe ratio (Exact SR), considered sample sizes (n), the simulated bias (Bias) and the mean square error (MSE) for the empirical Sharpe ratio and copula-based Sharpe ratio and the relative efficiency (RE) of two estimators. The results show that the MSE and bias in all cases decrease with the sample size, as expected. In all cases, the relative efficiency of the copula-based Sharpe ratio compared to the empirical method has increased significantly when the sample size increases. The only exception is the case that the marginal distributions are normal, as we see in Table 1. It can be seen that when the marginal distributions go towards skewness, the copula-based method will have a much better performance than the empirical method.

By comparing tables 1, 4, and 6, it can be observed that when the marginal distributions are symmetric and heavy-tailed, the efficiency is greater than when they are normal. When one of the marginal distributions deviates from symmetry and becomes skewed, the accuracy of the copula-based estimation of the Sharpe ratio increases compared to the empirical method (tables 7, 8). In this case, the larger weight of the asset with a skewed distribution assigned the higher accuracy of the copula-based estimation compared to the empirical method (tables 3, 5). Finally, the highest efficiency is achieved when both assets have a skewed distribution. In this case, the empirical method for estimating the Sharpe ratio performs very poorly compared to the Copula-based method (table 2).

In all cases, we see a smaller relative efficiency for the strong positive dependency (in terms of Kendall's tau) and a larger relative efficiency for the strong negative dependency. The highest efficiency was achieved in $n = 500$ at $\tau = 0.3$ with an upper-tailed dependent copula structure when both marginal distributions are skewed. The lowest efficiency was also obtained in $n = 50$ at $\tau = 0.8$ with the symmetric copula structure when both marginal distributions are normal.

In sum, when the assets are normally distributed, using the copula-based method or empirical method to estimate the Sharpe ratio does not make much difference. Financial data typically don't follow a normal distribution. So, using empirical methods for estimation isn't accurate and reduces the precision of the

Table 1: Simulation results when marginal distributions are $X \sim normal(1, 2)$ and $Y \sim normal(1, 2)$

Model	Kendall's τ	Exact SR	n	Empirical SR		Copula-based SR		RE
				Bias	MSE	Bias	MSE	
Clayton	0.8	0.7199	50	0.0174	0.0311	0.0252	0.0320	0.9709
			200	0.0007	0.0065	0.0026	0.0066	0.9953
			500	0.0004	0.0029	0.0012	0.0029	1.0007
	0.3	0.8051	50	0.0149	0.0342	0.0242	0.0349	0.9795
			200	0.0048	0.0086	0.0073	0.0086	0.9991
			500	0.0005	0.0030	0.0014	0.0030	1.0022
	-0.3	1.1222	50	0.0132	0.0261	0.0229	0.0261	0.9988
			200	0.0013	0.0067	0.0036	0.0066	1.0115
			500	-0.0011	0.0024	0.0003	0.0024	1.0152
Frank	0.8	0.7200	50	0.0113	0.0241	0.0194	0.0250	0.9636
			200	0.0047	0.0062	0.0066	0.0063	0.9892
			500	0.0001	0.0024	0.0009	0.0024	0.9926
	0.3	0.8136	50	0.0108	0.0265	0.0198	0.0275	0.9640
			200	0.0030	0.0068	0.0052	0.0068	0.9899
			500	0.0002	0.0027	0.0010	0.0027	0.9994
	-0.3	1.1116	50	0.0154	0.0395	0.0227	0.0390	1.0130
			200	0.0071	0.0089	0.0087	0.0088	1.0148
			500	-0.0018	0.0035	-0.0008	0.0034	1.0312
Gumbel	0.8	0.7153	50	0.0086	0.0257	0.0160	0.0264	0.9730
			200	0.0031	0.0064	0.0049	0.0064	0.9934
			500	-0.0005	0.0026	0.0002	0.0026	0.9988
	0.3	0.8050	50	0.0192	0.0263	0.0277	0.0271	0.9739
			200	0.0043	0.0060	0.0061	0.0060	0.9940
			500	0.0018	0.0023	0.0026	0.0023	1.0037

estimates. It is recommended to use copula-based methods for this type of data, as they provide more reliable results and involve less error.

Three box plots are used to provide an overall picture of the performance of the two estimation methods of the Sharpe ratio - copula-based and empirical. The box plot of Figures 1, 2, and 3 is drawn for the Clayton copula with the marginal distributions (normal, normal), (log-normal, log-normal), and (logistic, log-normal). The true value of the Sharpe ratio is indicated by a horizontal dashed line in each figure. Figure 1 shows that for (normal, normal) marginal distributions, the performance of the two measures is very close to each other. According to figures 2 and 3, which are plotted for the marginal distributions of (logistic, log-normal) and (log-normal, log-normal), respectively, it is observed that the copula-based method performs significantly better than the empirical method. On the other hand, in all the box plots, it can be observed that as the sample size increases, the estimated values get closer to the true value, thereby reducing the bias.

Table 2: Simulation results when marginal distributions are $X \sim \log normal(-0.35, 0.84)$ and $Y \sim \log normal(-0.35, 0.84)$.

Model	Kendall's τ	Exact SR	n	Empirical SR		Copula-based SR		RE
				Bias	MSE	Bias	MSE	
Clayton	0.8	0.7954	50	0.0965	0.0367	0.0197	0.0150	2.4487
			200	0.0469	0.0129	0.0097	0.0036	3.6140
			500	0.0250	0.0063	0.0025	0.0013	4.8006
	0.3	0.8840	50	0.1268	0.0614	0.0182	0.0208	2.9558
			200	0.0564	0.0199	0.0055	0.0050	3.9676
			500	0.0255	0.0111	-0.0001	0.0019	5.9852
	-0.3	0.9722	50	0.2014	0.1147	0.0313	0.0281	4.0822
			200	0.0769	0.0367	0.0085	0.0069	5.3198
			500	0.0416	0.0196	0.0041	0.0029	6.8139
Frank	0.8	0.7720	50	0.1029	0.0410	0.0270	0.0134	3.0495
			200	0.0338	0.0118	0.0017	0.0030	3.9654
			500	0.0179	0.0065	0.0012	0.0012	5.2574
	0.3	0.8641	50	0.1272	0.0517	0.0153	0.0152	3.3941
			200	0.0467	0.0179	0.0038	0.0038	4.6683
			500	0.0194	0.0101	0.0015	0.0016	6.3267
	-0.3	0.9925	50	0.2152	0.1284	0.0265	0.0306	4.2039
			200	0.0948	0.0425	0.0047	0.0079	5.3781
			500	0.0416	0.0211	-0.0018	0.0030	7.0598
Gumbel	0.8	0.7110	50	0.1471	0.0619	0.0299	0.0171	3.6094
			200	0.0652	0.0190	0.0072	0.0035	5.4550
			500	0.0354	0.0106	0.0036	0.0014	7.4488
	0.3	0.7789	50	0.1609	0.0746	0.0230	0.0161	4.6288
			200	0.0811	0.0269	0.0102	0.0039	6.8833
			500	0.0365	0.0134	0.0021	0.0016	8.5702

Table 3: Simulation results when marginal distributions are $X \sim normal(1, 2)$ and $Y \sim \log norm(-0.35, 0.84)$.

Model	Kendall's τ	Exact SR	n	Empirical SR		Copula-based SR		RE
				Bias	MSE	Bias	MSE	
Clayton	0.8	0.7785	50	0.0609	0.0267	0.0080	0.0150	1.7775
			200	0.0313	0.0099	0.0043	0.0034	2.9124
			500	0.0146	0.0055	0.0023	0.0015	3.6740
	0.3	0.8576	50	0.0980	0.0450	0.0240	0.0222	2.0300
			200	0.0471	0.0161	0.0061	0.0052	3.0727
			500	0.0233	0.0091	0.0018	0.0020	4.5372
	-0.3	1.0286	50	0.2320	0.1449	0.0472	0.0417	3.4752
			200	0.0872	0.0457	0.0064	0.0087	5.2593
			500	0.0510	0.0262	0.0051	0.0044	5.9984
Frank	0.8	0.7682	50	0.0710	0.0262	0.0161	0.0142	1.8490
			200	0.0288	0.0099	0.0022	0.0032	3.0653
			500	0.0109	0.0049	0.0008	0.0012	3.9560
	0.3	0.8458	50	0.1070	0.0416	0.0246	0.0172	2.4210
			200	0.0400	0.0160	0.0067	0.0043	3.7219
			500	0.0239	0.0081	0.0017	0.0017	4.7353
	-0.3	1.0413	50	0.2192	0.1591	0.0370	0.0499	3.1892
			200	0.0980	0.0484	0.0135	0.0113	4.2756
			500	0.0575	0.0244	0.0073	0.0047	5.1388
Gumbel	0.8	0.7531	50	0.0748	0.0293	0.0160	0.0143	2.0476
			200	0.0319	0.0095	0.0030	0.0031	3.0810
			500	0.0161	0.0050	0.0021	0.0012	4.2290
	0.3	0.8145	50	0.1182	0.0421	0.0268	0.0173	2.4284
			200	0.0414	0.0164	0.0064	0.0038	4.3246
			500	0.0228	0.0077	-0.0016	0.0016	4.7473

Table 4: Simulation results when marginal distributions are $X \sim logistic(1, 1.1)$ and $Y \sim logistic(1, 1.1)$.

Model	Kendall's τ	Exact SR	n	Empirical SR		Copula-based SR		RE
				Bias	MSE	Bias	MSE	
Clayton	0.8	0.7227	50	0.0134	0.0300	0.0164	0.0272	1.1025
			200	0.0023	0.0065	0.0038	0.0059	1.1062
			500	0.0010	0.0031	0.0011	0.0027	1.1569
	0.3	0.8045	50	0.0216	0.0380	0.0222	0.0343	1.0815
			200	0.0081	0.0090	0.0077	0.0081	1.1098
			500	0.0027	0.0037	0.0034	0.0032	1.1606
	-0.3	1.1188	50	0.0240	0.0322	0.0256	0.0290	1.1088
			200	0.0067	0.0072	0.0056	0.0065	1.1143
			500	0.0022	0.0033	0.0026	0.0029	1.1654
Frank	0.8	0.7244	50	0.0149	0.0100	0.0145	0.0257	1.0142
			200	0.0031	0.0064	0.0051	0.0063	1.0199
			500	0.0002	0.0026	0.0009	0.0024	1.0740
	0.3	0.8179	50	0.0124	0.0292	0.0145	0.0277	1.0552
			200	0.0089	0.0074	0.0096	0.0070	1.0604
			500	0.0020	0.0029	0.0026	0.0027	1.0756
	-0.3	1.1009	50	0.0189	0.0397	0.0226	0.0357	1.1135
			200	0.0033	0.0106	0.0020	0.0088	1.2114
			500	0.0008	0.0044	-0.0005	0.0036	1.2317
Gumbel	0.8	0.7157	50	0.0179	0.0298	0.0184	0.0278	1.0722
			200	0.0059	0.0071	0.0055	0.0065	1.0839
			500	0.0015	0.0026	0.0015	0.0024	1.1002
	0.3	0.8042	50	0.0142	0.0279	0.0205	0.0258	1.0786
			200	0.0045	0.0064	0.0041	0.0059	1.0872
			500	0.0020	0.0027	0.0019	0.0024	1.1042

Table 5: Simulation results when marginal distributions are $X \sim logistic(1, 1.1)$ and $Y \sim \log normal(-0.35, 0.84)$.

Model	Kendall's τ	Exact SR	n	Empirical SR		Copula-based SR		RE
				Bias	MSE	Bias	MSE	
Clayton	0.8	0.7803	50	0.0724	0.0282	0.0175	0.0166	1.7053
			200	0.0325	0.0096	0.0043	0.0035	2.7230
			500	0.0161	0.0065	0.0023	0.0016	4.1563
	0.3	0.8587	50	0.1054	0.0449	0.0272	0.0231	1.9461
			200	0.0403	0.0169	0.0063	0.0052	3.2663
			500	0.0236	0.0087	0.0006	0.0020	4.3856
	-0.3	1.0310	50	0.2329	0.1468	0.0452	0.0410	3.5828
			200	0.1026	0.0511	0.0114	0.0098	5.2097
			500	0.0588	0.0238	0.0023	0.0040	5.9223
Frank	0.8	0.7687	50	0.0683	0.0274	0.0106	0.0138	1.9834
			200	0.0322	0.0095	0.0033	0.0031	3.0677
			500	0.0106	0.0057	-0.0011	0.0012	4.6208
	0.3	0.8474	50	0.0939	0.0379	0.0131	0.0174	2.1790
			200	0.0463	0.0170	0.0049	0.0043	4.0000
			500	0.0265	0.0083	0.0026	0.0017	4.8709
	-0.3	1.0384	50	0.2412	0.1571	0.0520	0.0475	3.3104
			200	0.1001	0.0527	0.0103	0.0108	4.8675
			500	0.0511	0.0264	0.0056	0.0049	5.3951
Gumbel	0.8	0.7484	50	0.0821	0.0312	0.0159	0.0147	2.1240
			200	0.0355	0.0096	0.0029	0.0031	3.0860
			500	0.0145	0.0059	0.0022	0.0012	4.7467
	0.3	0.8104	50	0.1079	0.0485	0.0192	0.0177	2.7356
			200	0.0419	0.0166	0.0037	0.0039	4.2918
			500	0.0148	0.0085	0.0012	0.0015	5.5455

Table 6: Simulation results when marginal distributions are $X \sim logistic(1, 1.1)$ and $Y \sim normal(1, 2)$.

Model	Kendall's τ	Exact SR	n	Empirical SR		Copula-based SR		RE
				Bias	MSE	Bias	MSE	
Clayton	0.8	0.7217	50	0.0211	0.0268	0.0271	0.0271	0.9859
			200	0.0059	0.0068	0.0076	0.0066	1.0267
			500	0.0016	0.0026	0.0022	0.0025	1.0306
	0.3	0.8050	50	0.0129	0.0337	0.0201	0.0338	0.9988
			200	0.0055	0.0083	0.0068	0.0080	1.0414
			500	0.0024	0.0030	0.0030	0.0029	1.0467
	-0.3	1.1203	50	0.0229	0.0267	0.0303	0.0266	1.0001
			200	0.0059	0.0066	0.0083	0.0063	1.0436
			500	0.0014	0.0027	0.0017	0.0026	1.0492
Frank	0.8	0.7224	50	0.0202	0.0263	0.0271	0.0271	0.9722
			200	0.0057	0.0063	0.0074	0.0063	0.9937
			500	-0.0009	0.0025	-0.0002	0.0024	1.0073
	0.3	0.8158	50	0.0176	0.0271	0.0248	0.0278	0.9755
			200	0.0023	0.0064	0.0045	0.0064	1.0054
			500	-0.0006	0.0027	0.0003	0.0027	1.0115
	-0.3	1.1061	50	0.0229	0.0426	0.0324	0.0420	1.0144
			200	0.0031	0.0085	0.0054	0.0081	1.0481
			500	0.0023	0.0034	0.0027	0.0033	1.0509
Gumbel	0.8	0.7160	50	0.0186	0.0275	0.0250	0.0275	1.0009
			200	0.0022	0.0064	0.0033	0.0063	1.0177
			500	0.0004	0.0025	0.0008	0.0024	1.0251
	0.3	0.8048	50	0.0107	0.0254	0.0185	0.0253	1.0016
			200	0.0049	0.0062	0.0069	0.0061	1.0194
			500	0.0003	0.0025	0.0010	0.0024	1.0263

Table 7: Simulation results when marginal distributions are $X \sim \log normal(-0.35, 0.84)$ and $Y \sim normal(1, 2)$.

Model	Kendall's τ	Exact SR	n	Empirical SR		Copula-based SR		RE
				Bias	MSE	Bias	MSE	
Clayton	0.8	0.7785	50	0.0156	0.0203	0.0158	0.0198	1.0254
			200	0.0084	0.0055	0.0033	0.0053	1.0337
			500	0.0037	0.0022	0.0022	0.0020	1.0907
	0.3	0.8576	50	0.0231	0.0313	0.0203	0.0301	1.0382
			200	0.0076	0.0069	0.0060	0.0063	1.0846
			500	0.0027	0.0029	0.0018	0.0026	1.1252
	-0.3	1.0286	50	0.0140	0.0254	0.0157	0.0216	1.1720
			200	0.0106	0.0074	0.0080	0.0060	1.2327
			500	0.0038	0.0030	0.0014	0.0024	1.2851
Frank	0.8	0.7682	50	0.0105	0.0200	0.0066	0.0188	1.0631
			200	0.0040	0.0051	0.0028	0.0046	1.1225
			500	0.0021	0.0020	0.0007	0.0017	1.1921
	0.3	0.8458	50	0.0204	0.0270	0.0156	0.0241	1.1194
			200	0.0028	0.0064	0.0017	0.0056	1.1614
			500	-0.0004	0.0026	-0.0004	0.0021	1.2507
	-0.3	1.0413	50	0.0332	0.0364	0.0244	0.0300	1.2166
			200	0.0085	0.0091	0.0051	0.0072	1.2589
			500	0.0031	0.0035	0.0016	0.0027	1.3352
Gumbel	0.8	0.7531	50	0.0243	0.0207	0.0209	0.0187	1.1057
			200	0.0094	0.0054	0.0058	0.0045	1.2186
			500	0.0008	0.0024	0.0010	0.0018	1.3698
	0.3	0.8144	50	0.0281	0.0261	0.0184	0.0216	1.2093
			200	0.0087	0.0062	0.0050	0.0045	1.3702
			500	0.0028	0.0028	0.0017	0.0019	1.4271

Table 8: Simulation results when marginal distributions are $X \sim \log normal(-0.35, 0.84)$ and $Y \sim logistic(1, 1.1)$.

Model	Kendall's τ	Exact SR	n	Empirical SR		Copula-based SR		RE
				Bias	MSE	Bias	MSE	
Clayton	0.8	0.7803	50	0.0123	0.0253	0.0116	0.0222	1.1421
			200	0.0074	0.0060	0.0054	0.0052	1.1582
			500	0.0011	0.0025	0.0002	0.0022	1.1593
	0.3	0.8587	50	0.0216	0.0310	0.0133	0.0266	1.1682
			200	0.0025	0.0077	0.0056	0.0063	1.2145
			500	0.0019	0.0033	0.0011	0.0028	1.3302
	-0.3	1.0310	50	0.0299	0.0328	0.0197	0.0279	1.1748
			200	0.0089	0.0081	0.0029	0.0063	1.2931
			500	0.0040	0.0035	0.0005	0.0024	1.4539
Frank	0.8	0.7687	50	0.0203	0.0243	0.0152	0.0218	1.1527
			200	0.0058	0.0053	0.0044	0.0045	1.1650
			500	-0.0005	0.0025	-0.0016	0.0020	1.2466
	0.3	0.8474	50	0.0245	0.0301	0.0154	0.0241	1.2497
			200	0.0074	0.0066	0.0057	0.0053	1.2509
			500	0.0021	0.0030	0.0017	0.0022	1.3370
	-0.3	1.0384	50	0.0306	0.0397	0.0148	0.0300	1.3252
			200	0.0104	0.0103	0.0026	0.0074	1.3986
			500	0.0036	0.0044	0.0002	0.0029	1.4896
Gumbel	0.8	0.7484	50	0.0291	0.0240	0.0192	0.0207	1.1567
			200	0.0097	0.0060	0.0041	0.0045	1.3200
			500	0.0044	0.0025	0.0019	0.0017	1.4600
	0.3	0.8104	50	0.0304	0.0275	0.0176	0.0211	1.3067
			200	0.0101	0.0076	0.0039	0.0048	1.5834
			500	0.0043	0.0032	0.0015	0.0020	1.6035

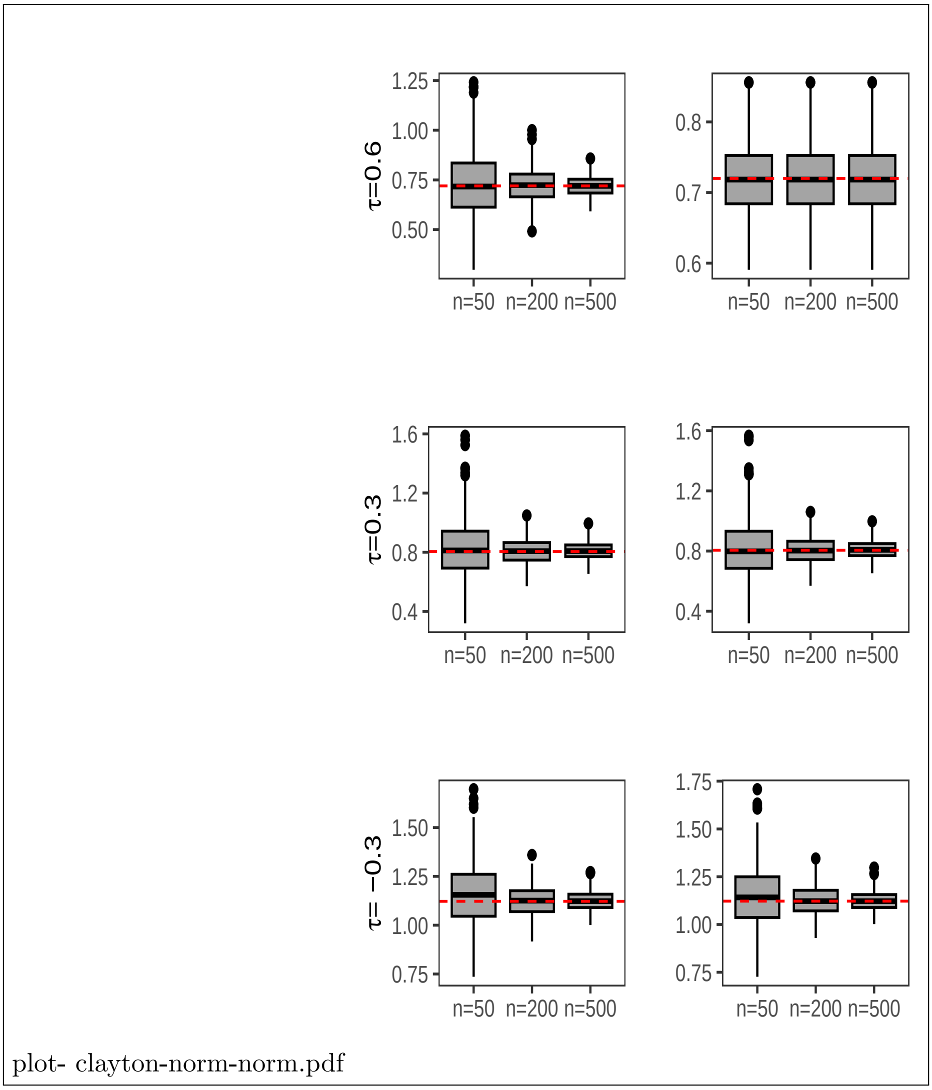


Figure 1: The box plot of estimates of the Sharpe ratio using the copula-based method (left panel) and the empirical method (right panel) for the Clayton copula and normal marginal distributions.

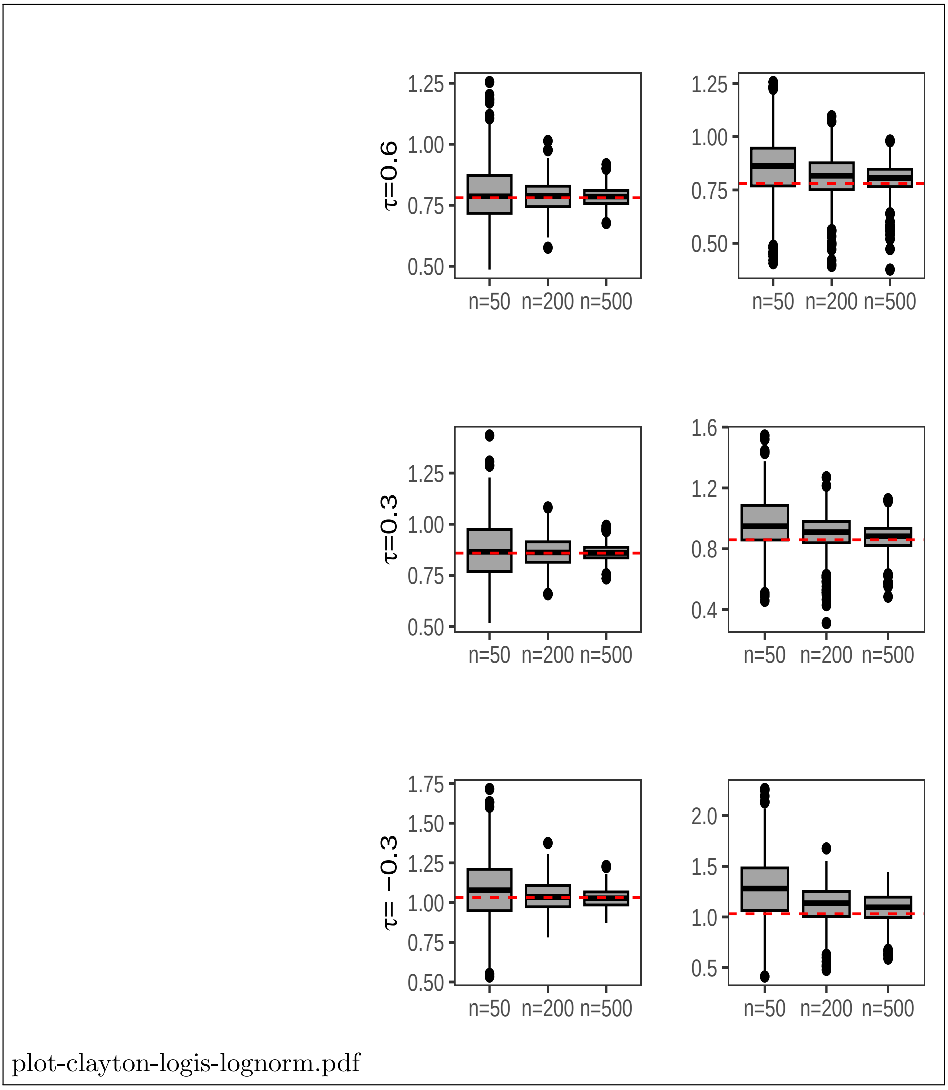


Figure 2: The box plot of estimates of the Sharpe ratio using the copula-based method (left panel) and the empirical method (right panel) for the Clayton copula and logistic and lognormal marginal distributions.

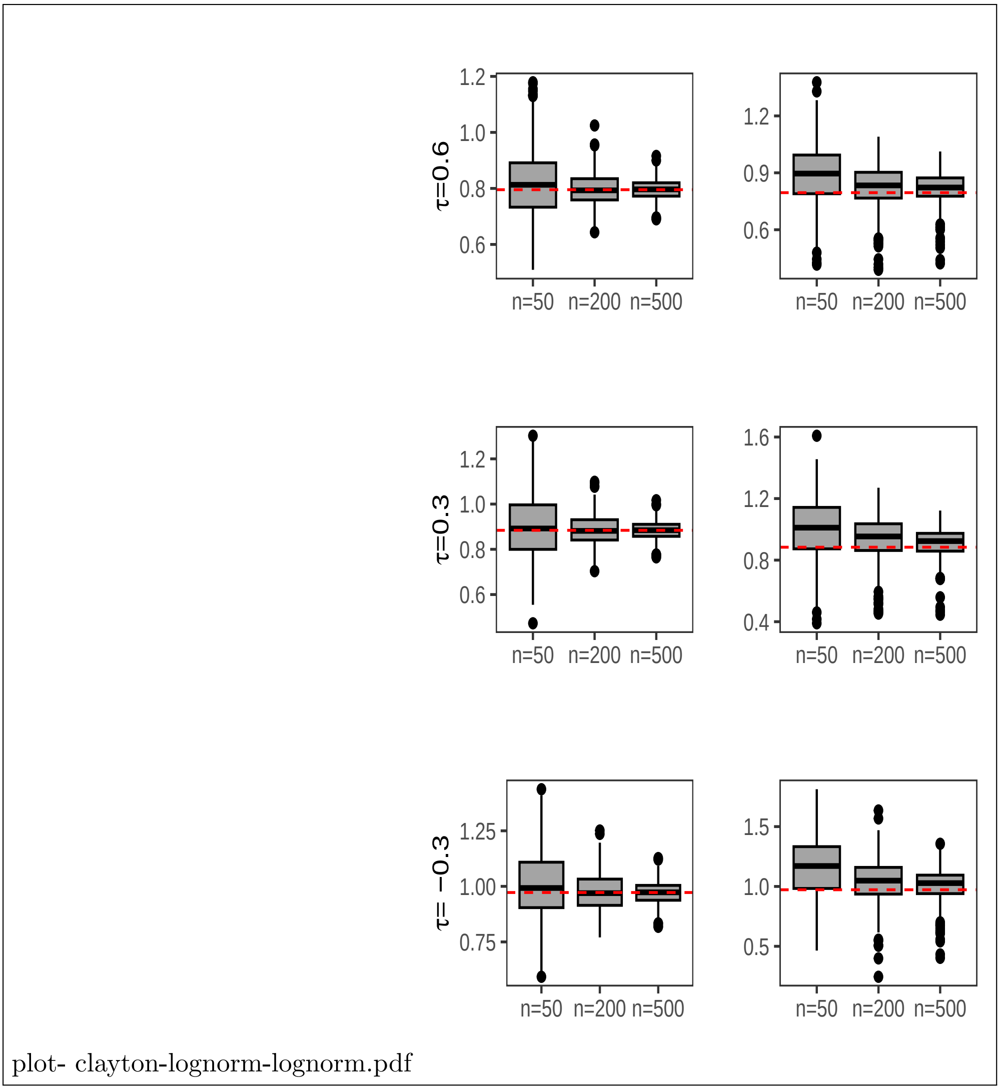


Figure 3: The box plot of estimates of the Sharpe ratio using the copula-based method (left panel) and the empirical method (right panel) for the Clayton copula and lognormal marginal distributions.

Table 9: Kendall's τ between CHF/USD, EUR/USD, and GBP/USD

	CHF/USD	EUR/USD	GBP/USD
CHF/USD	1.000	0.5371	0.4485
EUR/USD	0.5371	1.0000	0.6035
GBP/USD	0.4485	0.6035	1.0000

6. Data Analysis

In this section, we compare the copula approach and the empirical method to calculate the Sharpe ratio of a portfolio using real data sets. Two cases where assets have positive or negative dependence are considered. To compare, the value of the Sharpe ratio has been calculated under the assumption of independence of assets, too.

6.1 Portfolios with positive dependent assets

We use three European exchange rates, GBP/USD, EUR/USD, and CHF/USD, and then create four portfolios including them. The first portfolio is composed of exchange rate GBP against USD and exchange rate EUR against USD, denoted by GBP/USD, EUR/USD. The second portfolio is composed of CHF/USD, EUR/USD, the third portfolio is CHF/USD, GBP/USD, and the fourth portfolio is composed of all three exchange rates. The analyzed period was from the 1st of July 2010 to the 1st of June 2021 in monthly frequency. Data were selected from finance.yahoo.com. Table 9 shows the values of Kendall's τ between the pairs of exchange rates that indicate positive dependence. Usually, stock returns are time series data. Suitable time series models to fit returns are indicated in table 10. GBP and CHF returns have a better fit to AR(1), and EUR returns have a better fit to ARMA(1,1). To calculate the Sharpe ratio using the copula approach, first, suitable marginal distributions for returns and then an appropriate copula structure are selected for each pair of portfolio components. The Sharpe ratio is then calculated by the formula (2.1). For given time series data, after applying suitable time series models, marginal parameters were obtained using the ARMA-filtered residuals. Different distributions were fitted to the ARMA-filtered residuals, and the normal, logistic, Student-t, and Cauchy distributions seemed to be suitable. Table 11 shows the estimated parameters, the Kolmogorov-Smirnov (K-S) statistic, the Anderson-Darling (A-D) statistic, Akaike's Information Criterion (AIC), Bayesian Information Criterion (BIC), and p-values of the fitted distributions. According to the table, we choose the logistic distribution for GBP/USD and CHF/USD returns and the Student-t distribution for EUR/USD returns.

Table 10: Time Series Models of Returns

Return	Time Series Model	Model Equation
GBP	AR(1)	$x_t = 0.0001 + 0.0871x_{t-1} + \epsilon_t$
EUR	ARMA(1,1)	$x_t = -0.0002 - 0.9395x_{t-1} + \epsilon_t - 0.8798\epsilon_{t-1}$
CHF	AR(1)	$x_t = 0.0014 - 0.1478x_{t-1} + \epsilon_t$

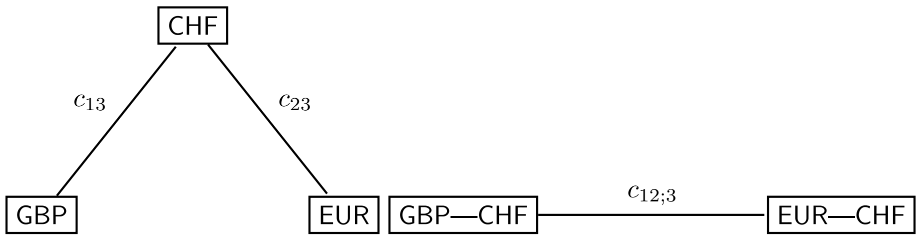


Figure 4: C- vine structures of the three pair copula constructions.

The copula selection is performed using the BiCopSelect function in the R package VineCopula Schepsmeier *et al.* (2018). Table 12 shows the estimated dependency parameter, the values of Kendall's tau, the Cramér–von Mises S_n statistic Genest *et al.* (2009), AIC, BIC, and p-value for 5 copulas. The Frank copula is a good fit for the dependence structure of portfolios 1 and 3 and the Student-t copula is a good fit for portfolio 2. Using the bivariate copula approach, we can construct the joint distribution of variables with a dependence structure characterized. Vine copulas build a multivariate copula using bivariate copulas. For three dimensions:

$$f(x_1, x_2, x_3) = c_{12;3}(F_{1|3}(x_1|x_3), F_{2|3}(x_2|x_3); x_3) \times c_{13}(F_1(x_1), F_3(x_3)) \\ \times c_{23}(F_2(x_2), F_3(x_3)) f_3(x_3) f_2(x_2) f_1(x_1).$$

Figure 4 indicates the graphical presentation of the three-dimensional vine copula including GBP, EUR, and CHF. In figure 5, deviation from the Gaussian copula can be seen in all three panels. The lower right panel has upper and lower tail dependence. The lower left panel has no tail dependence and is approximately symmetric. Copula's parameter estimation results are contained in the table 13.

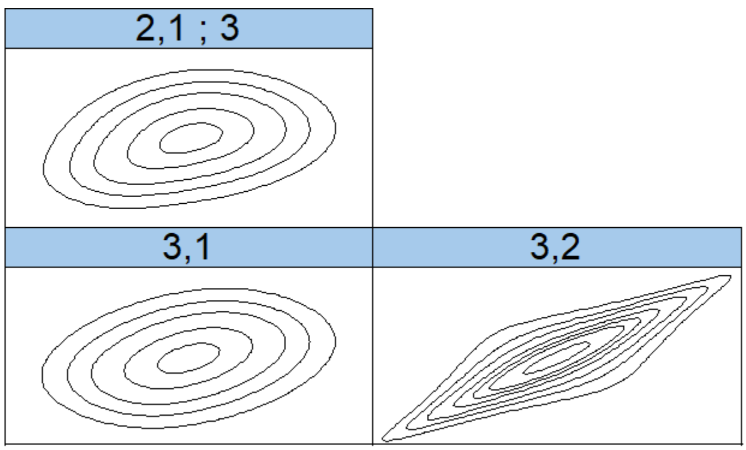
Table 14 shows the value of the Sharpe ratio calculated by three methods: empirical, copula-based, and under the assumption of independence of returns of the portfolio assets. In this table, portfolio 1 includes GBP and EUR, portfolio 2 includes CHF and EUR, portfolio 3 includes CHF and GBP, and portfolio 4 includes GBP, EUR, and CHF. As it can be seen, due to the positive dependence of assets, the Sharpe ratio calculated under the assumption of independence of assets is overestimated. In most cases, financial data affect each other and are interdependent.

Table 11: Fitted marginal distributions for ARMA-filtered residuals of exchange rate returns

Asset	Distribution	Estimated parameters	K-S	A-D	AIC	BIC	P-Value
GBP/USD	Normal	$\hat{\mu} = -0/0037(0.087), \hat{\sigma} = 0/999(0.061)$	0.065	0.439	378.48	384.25	0.61
	Logistic	**Location= 0/0126 (0.083) , Scale = 0/551 (0.039)**	**0.054**	**0.316**	**374.01**	**379.78**	**0.83**
	t	df = 7/76 (0.044)	0.055	0.322	375.67	384.32	0.80
	Gumbel	Location= -0.5069 (0.057) , Scale = 1.075 (0.049)	0.121	3.273	410.68	416.46	0.04
EUR/USD	Normal	$\hat{\mu} = -0/0010(0.016), \hat{\sigma} = 0/188(0.011)$	0.053	0.422	-62.49	-56.72	0.85
	Logistic	Location=0/0031 (0.015), Scale =0/105 (0.007)	0.046	0.297	-61.02	-59.26	0.92
	t	**df =8/95 (0.062)**	**0.045**	**0.278**	**-63.15**	**-64.44**	**0.95**
	Gumbel	Location= -0.0971 (0.041) , Scale = 0.203 (0.067)	0.124	3.443	-27.90	-22.13	0.04
CHF/USD	Normal	$\hat{\mu} = 0/0080(0.087), \hat{\sigma} = 1/000(0.061)$	0.058	0.515	378.63	384.40	0.77
	Logistic	**Location=0/0118 (0.082), Scale = 0/547 (0.039)**	**0.050**	**0.240**	**373.16**	**378.92**	**0.90**
	t	df = 6/38 (0.032)	0.053	0.243	375.05	383.70	0.85
	Gumbel	Location= -0.5014 (0.027) , Scale = 1.125 (0.051)	0.109	4.001	418.62	424.39	0.08

Table 12: The estimated parameter and copula goodness-of-fit test statistic for three portfolios

Portfolio	Assumed copula	θ	log-likelihood	S_n Statistic	AIC	P-Value
rGBP/USD rEUR/USD	Student-t	0.38 (0.080), df= 6.68 (0.049)	8.90	0.020	-13.84	0.49
	Gumbel	1.30 (0.079)	8.45	0.018	-14.90	0.63
	Clayton	0.66 (0.020)	4.13	0.046	-9.04	0.00
	Frank	**2.47 (0.032)**	**9.71**	**0.017**	**-17.42**	**0.69**
	Normal	0/35 (0.075)	7.90	0.022	-13.80	0.46
rCHF/USD -rEUR/USD	**Student-t**	**0.80 (0.041), df= 2 (0.036)**	**73.64**	**0.017**	**-143.28**	**0.49**
	Gumbel	2.30 (0.104)	56.37	0.021	-110.74	0.19
	Clayton	2.75 (0.208)	39.46	0.061	-76.91	0.00
	Frank	7.75 (0.058)	56.50	0.018	-110.99	0.46
	Normal	0.70 (0.036)	42.76	0.019	-83.52	0.35
rCHF/USD rGBP/USD	Student-t	0.30 (0.048), df= 6.77 (0.051)	5.13	0.016	-6.26	0.83
	Gumbel	1.21 (0.068)	4.56	0.017	-7.13	0.75
	Clayton	0.48 (0.142)	3.31	0.028	-4.62	0.17
	Frank	**1.81 (0.052)**	**5.41**	**0.014**	**-8.83**	**0.89**
	Normal	0.27 (0.082)	4.27	0.016	-6.55	0.78



vine copula.png

Figure 5: Contour plots. Respectively, Survival BB8, Frank, and t copulas.

Table 13: Selected copula families in vine copula

Term	Family copula	Parameter(s)	Kendall's τ
c_{13}	Frank	1.81	0.19
c_{23}	t	0.81, df = 2	0.6
$c_{12;3}$	Survival BB8	1.87, 0.82	0.19

Table 14: Estimated value of the Sharpe ratio for four portfolios

Portfolio	Weight	Independence	Empirical method	Copula method
Portfolio (1)	0.2	0.0116	-0.0006	0.0109
	0.4	0.0162	-0.0002	0.0142
	0.6	0.0199	0.0005	0.0172
	0.8	0.0206	0.0022	0.0189
Portfolio (2)	0.2	0.0234	0.0016	0.0199
	0.4	0.0422	0.0055	0.0321
	0.6	0.0572	0.0128	0.0437
	0.8	0.0630	0.0283	0.0541
Portfolio (3)	0.2	0.0370	0.0190	0.0345
	0.4	0.0549	0.0328	0.0489
	0.6	0.0645	0.0427	0.0584
	0.8	0.0654	0.0477	0.0623
Portfolio (4)	$w_1 = 0.2, w_2 = 0.2$	0.0651	0.0239	0.0534
	$w_1 = 0.2, w_2 = 0.4$	0.0542	0.0090	0.0408
	$w_1 = 0.4, w_2 = 0.2$	0.0599	0.0175	0.0468
	$w_1 = 0.4, w_2 = 0.4$	0.0404	0.0049	0.0313

Also, Pearson correlation considers only linear dependence. Therefore, the value calculated with the copula method is a better estimate. According to this table, portfolio 1 has the lowest Sharpe ratio and the exchange rate of CHF increases the Sharpe ratio of the portfolio. Therefore, as the weight of CHF increases, the Sharpe ratio increases (In all three methods, portfolio 3 seems to have a larger Sharpe ratio). Also, the portfolio with a higher positive correlation between the exchange rates has a smaller Sharpe ratio.

6.2 Portfolios with negative dependent assets

In this section, we examine the effect of the negative correlation between assets on the copula-based Sharpe ratio compared to the independence and empirical methods. Liu, Chang, and ChuiLiu et al. (2016) examined the effectiveness of Gold and the US dollar (USD) as hedge assets against stock prices for seven developed markets. According to Liu et al. (2016), USD and Gold are hedge assets in some countries during normal market conditions. They show that USD/CAD is negatively correlated with the S&P/TSX 300 and USD/CAD is a hedge asset for the portfolio. Also, USD/CAD and Gold have a negative correlation. The weekly data of stock indices S&P/TSX 300 in Canada, exchange rate USD/CAD, and Gold are collected over the period 13 September 2021 - 3 September 2022. These data are obtained from https://www.tgju.org/.

Table 15 indicates Kendall's tau between data and table 16 indicates time series models fitted to returns. Table 17 shows the goodness-of-fit test statistics for

Table 15: Kendall's tau between S&P/TSX 300, USD/CAD, and Gold

	S&P/TSX 300	USD/CAD	Gold/CAD
S&P/TSX 300	1.000	-0.5288	0.8384
USD/CAD	-0.5288	1.0000	-0.3672
Gold/CAD	0.8384	-0.3672	1.0000

Table 16: Time series model of the Returns

Return	Time series model	Model
S&P/TSX 300	AR(1)	$x_t = -0.0011 + 0.1610x_{t-1} + \epsilon_t$
USD/CAD	AR(1)	$x_t = 0.0008 + 0.2182x_{t-1} + \epsilon_t$
Gold/CAD	AR(1)	$x_t = -0.0005 + 0.2106x_{t-1} + \epsilon_t$

marginal distributions of filtered residuals. The Gumbel distribution is selected for USD/CAD filtered residuals, the t-distribution for S&P/TSX 300 filtered residuals, and the logistic distribution for Gold filtered residuals.

The result of copula goodness-of-fit tests given in Table 18 suggest that the rotated 90 degree Joe copula is a good fit for returns of USD/CAD and S&P/TSX 300, also the Rotated 270-degree Joe copula Joe (2014) for the returns of USD/CAD and Gold/CAD. The estimated values of the Sharpe ratio calculated by three methods are given in Table 19. As we can see, under the assumption of independence of returns and the empirical method, the estimated value of the Sharpe ratios is less than the copula approach. Therefore, when faced with a negative dependence, the independence assumption and empirical method underestimate the Sharpe ratio.

7. Concluding Remarks

This paper considered a copula-based Sharpe ratio for a two-asset portfolio, enhancing the accuracy of its estimation. The traditional covariance calculation is replaced with Hoeffding's formula, allowing for the consideration of various dependencies without assuming linearity. The study examines features of the copula-based Sharpe ratio, establishing its upper and lower bounds and demonstrating when conventional methods may lead to overestimation or underestimation. Through simulations, the copula-based Sharpe ratio consistently outperforms the empirical Sharpe ratio, particularly for asymmetric marginal distributions. Analysis of actual data shows that all three estimation methods—assuming independence, the empirical method, and the copula-based method—yield the same asset ranking. However, in cases of both positive and negative asset dependencies,

Table 17: Goodness-of-fit test for marginal distributions of filtered residuals

Asset	Distribution	Estimated parameters	K-S	A-D	AIC	BIC	P-Value
rUSD/CAD	Normal	$\hat{\mu} = -0.0347(0.141), \hat{\sigma} = 1.000(0.100)$	0.150	0.751	145.90	149.72	0.19
	Logistic	Location= -0.0558 (0.044), Scale = 0.586 (0.067)	0.153	0.868	147.92	151.74	0.17
	t	df = 4 (0.017)	0.150	0.750	147.90	153.63	0.18
	Gumbel	**Location= -0.5172 (0.010) , Scale = 0.866 (0.015)**	**0.122**	**0.648**	**145.35**	**149.18**	**0.43**
rS&P/TSX 300	Normal	$\hat{\mu} = 0.0192(0.041), \hat{\sigma} = 0.999(0.099)$	0.116	0.823	145.88	149.70	0.48
	Logistic	Location=0.1110 (0.033), Scale =0.545 (0.064)	0.074	0.515	143.83	147.65	0.92
	t	**df =5.45 (0.044)**	**0.072**	**0.496**	**142.74**	**146.47**	**0.94**
	Gumbel	Location= -0.5229 (0.024) , Scale = 1.212 (0.031)	0.187	2.683	167.95	171.77	0.05
rGold/CAD	Normal	$\hat{\mu} = 0.0179(0.141), \hat{\sigma} = 1.000(0.100)$	0.090	0.251	145.90	149.73	0.77
	Logistic	**Location=0.0456 (0.037), Scale =0.560 (0.046)**	**0.062**	**0.196**	**145.71**	**149.53**	**0.98**
	t	df =14.21 (0.121)	0.075	0.208	147.63	153.37	0.91
	Gumbel	Location= -0.495 (0.041) , Scale = 1.065 (0.028)	0.154	1.328	158.49	162.32	0.17
	skew-normal	mean = 0.0134 (0.034), sd =0.999 (0.072), $\gamma = 0.879(0.142)$	0.082	0.224	148.52	149.90	0.86
	skew- t	mean = 0.01800 (0.045), sd = 1.001 (0.112) , df = 15.93 (0.078), $\gamma = 0.880(0.054)$	0.072	0.201	146.94	149.89	0.94

Table 18: The estimated parameter and goodness-of-fit test statistic of copulas for USD/CAD-rS&P/TSX 300 and USD/CAD-Gold/CAD

Portfolio	Assumed copula	θ	log-likelihood	S_n Statistic	AIC	P-Value
USD/CAD	Student-t	-0.67 (0.042), df= 2.02 (0.112)	17.23	0.037	-30.46	0.10
- S&P/TSX 300	Plackett	0.09 (0.054)	14.45	0.043	-26.89	0.04
	Clayton	-0.68 (0.092)	15.34	0.042	-27.46	0.03
	Frank	-5.27 (0.117)	13.11	0.049	-24.21	0.03
	Normal	-0.70 (0.060)	14.87	0.041	-27.74	0.06
	Rotated Joe 90 degrees	**-3.10 (0.018)**	**22.15**	**0.014**	**-42.30**	**0.37**
USD/CAD	Student-t	-0.33 (0.159) , df= 2.08 (0.115)	4.94	0.059	-5.89	0.00
- Gold/CAD	Plackett	0.33 (0.084)	2.89	0.065	-3.78	0.00
	Clayton	-0.16 (0.098)	2.07	0.061	-2.14	0.01
	Frank	-2.12 (0.051)	2.62	0.067	-3.23	0.00
	Normal	-0.41 (0.120)	3.56	0.063	-5.13	0.00
	Rotated Joe 270 degrees	**-1.94 (0.024)**	**10.25**	**0.016**	**-18.49**	**0.12**

Table 19: Estimated values of Sharpe ratio for the portfolios

Portfolio	Weight	Independence	Empirical method	Copula method
S&P/TSX 300	0.2	0.1118	0.1165	0.1500
-USD/CAD	0.4	0.1008	0.0081	0.1409
	0.6	0.0817	-0.0455	0.0970
	0.8	0.0667	-0.0636	0.0710
USD/CAD	0.2	0.0149	-0.0294	0.0158
- Gold/CAD	0.4	0.0360	-0.0001	0.0425
	0.6	0.0646	0.0558	0.0839
	0.8	0.0893	0.1032	0.1064

the empirical method underestimates the Sharpe ratio compared to the copula-based method. Theoretical results and analyses reveal that increased positive dependence among portfolio assets results in a lower Sharpe ratio, while negative dependence leads to a higher Sharpe ratio.

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